CHAPTER 2

TIME VALUE OF MONEY

Discussion

Your Friend take your opinion about a critical decision

1. Take $13.5 million now.
2. Take $25.5 million in 30 annual installments.

What is your advice ?

To solve this problem

$1000 \neq 1000$

1. Why \$1000 now ≠ \$1000 after many years ?
2. How could we representation the problem?
3. How could we compare the value of money at different point in time ?

Why $1000 now ≠ $1000 after many years

because it can earn more money over time (**earning power**).

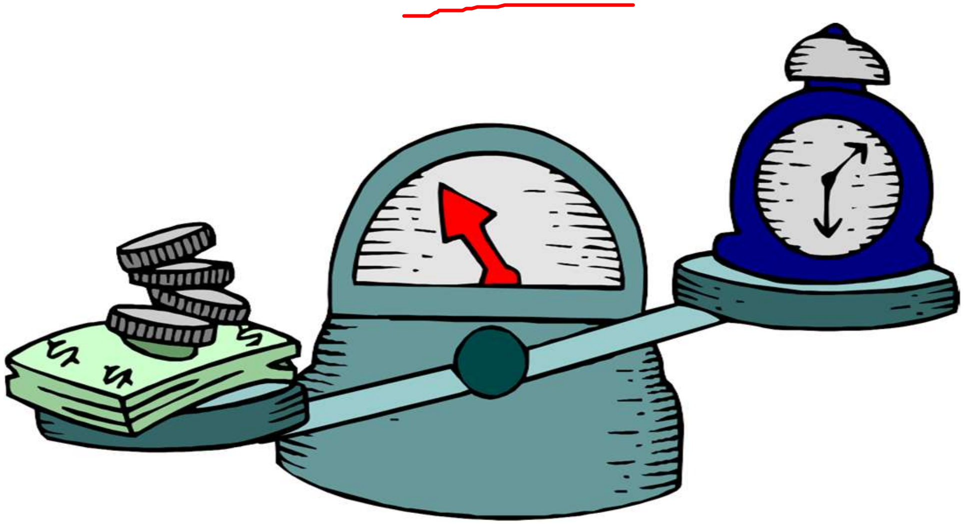
لأنها تستطيع أن تجني المزيد من المال بمرور الوقت (**القدرة على الكسب**).

because its purchasing power changes over time (**inflation**)

لأن قوتها الشرائية تتغير بمرور الوقت (**التضخم**).

Time value of money is measured in terms of (**interest rate**).

يتم قياس القيمة الزمنية للنقود من حيث (**سعر الفائدة**).



100 now

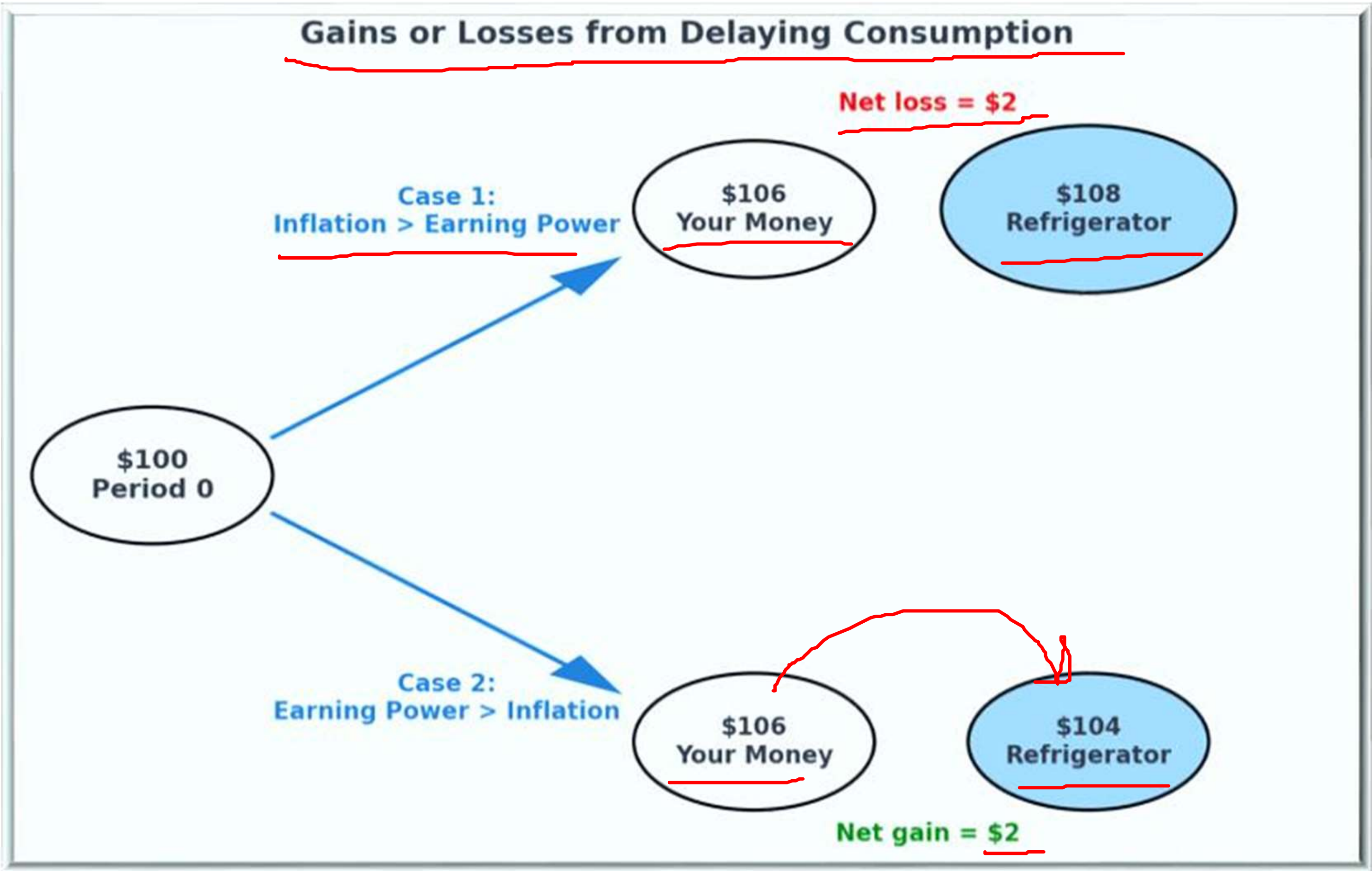
1

	Account Value 	Cost of Refrigerator 
Case 1: Inflation exceeds earning power	N = 0 \$100 N = 1 \$106 (earning rate =6%)	N = 0 \$100 N = 1 \$108 (inflation rate = 8%)
Case 2: Earning power exceeds inflation	N = 0 \$100 N = 1 \$106 (earning rate =6%)	N = 0 \$100 N = 1 \$104 (inflation rate = 4%)

Delaying Consumption

106 → res

106 → res



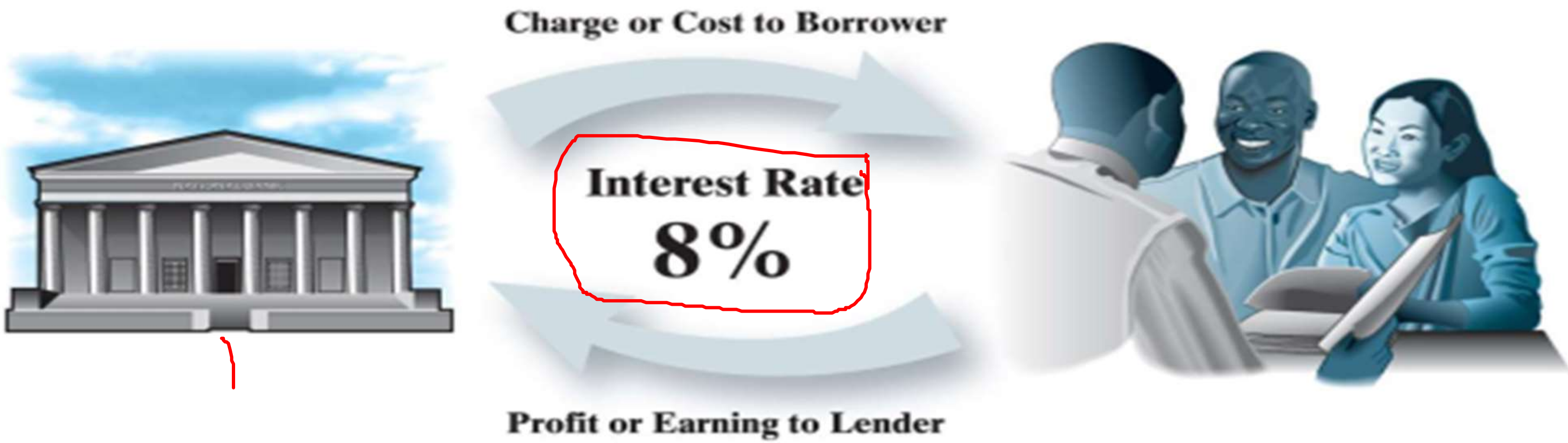
Interest Rate

The Cost of Money is established and measured by an **interest rate,** a percentage that is periodically applied and added to an amount of money over a specified length of time.

تكلفة المال : يتم تحديدها وقياسها من خلال **سعر الفائدة**
وهي نسبة مئوية يتم تطبيقها وإضافتها بشكل دوري إلى مبلغ من المال
على مدى فترة زمنية محددة.

Interest Rate

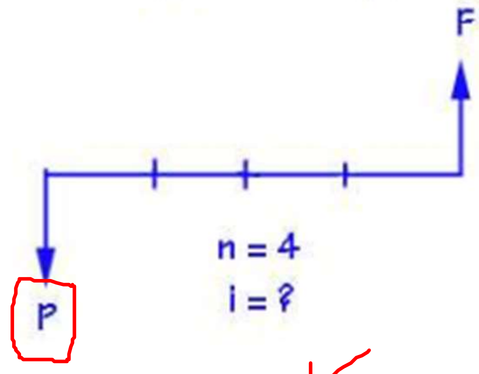
Interest is the cost of money—a cost to the borrower and an earning to the lender



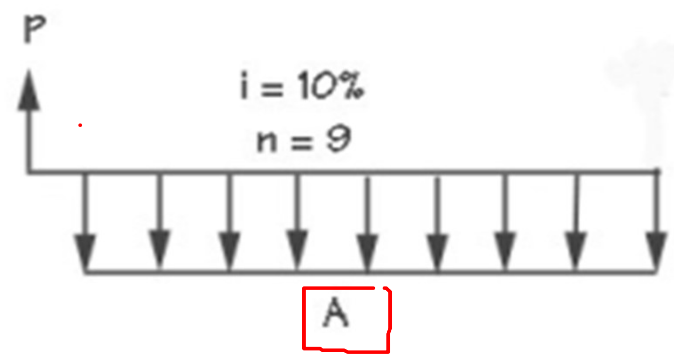
How could we representation the problem?

CASH FLOW DIAGRAMS

Cash Flow Diagram for Single Payment



Cash Flow Diagram for Uniform Series (A)



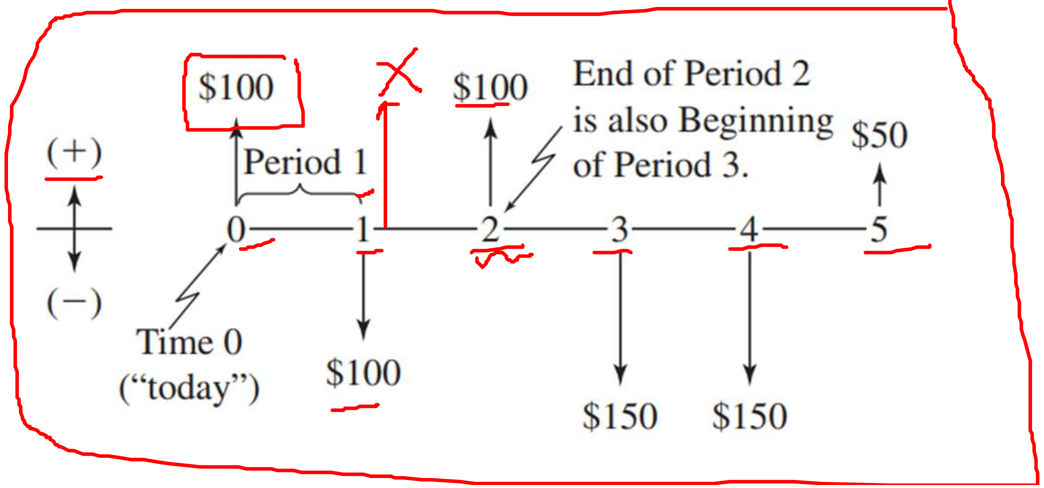
CASH FLOW DIAGRAMS

Cash Flow Diagram is dependent on Point of View

يعتمد مخطط التدفق النقدي على وجهة النظر

Net Cash Flow: A net cash flow is the difference between total cash receipts (inflows) and total cash disbursement (outflows) for a given period of time.

صافي التدفق النقدي: صافي التدفق النقدي هو الفرق بين إجمالي الإيرادات النقدية (التدفقات الداخلة) وإجمالي المدفوعات النقدية (التدفقات الخارجة) لفترة زمنية معينة.



Example

An investment of $10,000 can be made that will produce uniform annual revenue of $5,000 for 5 years and then have a salvage value of $2,000 at the end of 5 years. Annual expenses will be $3,000 at the end of each year for operating and maintaining the project.

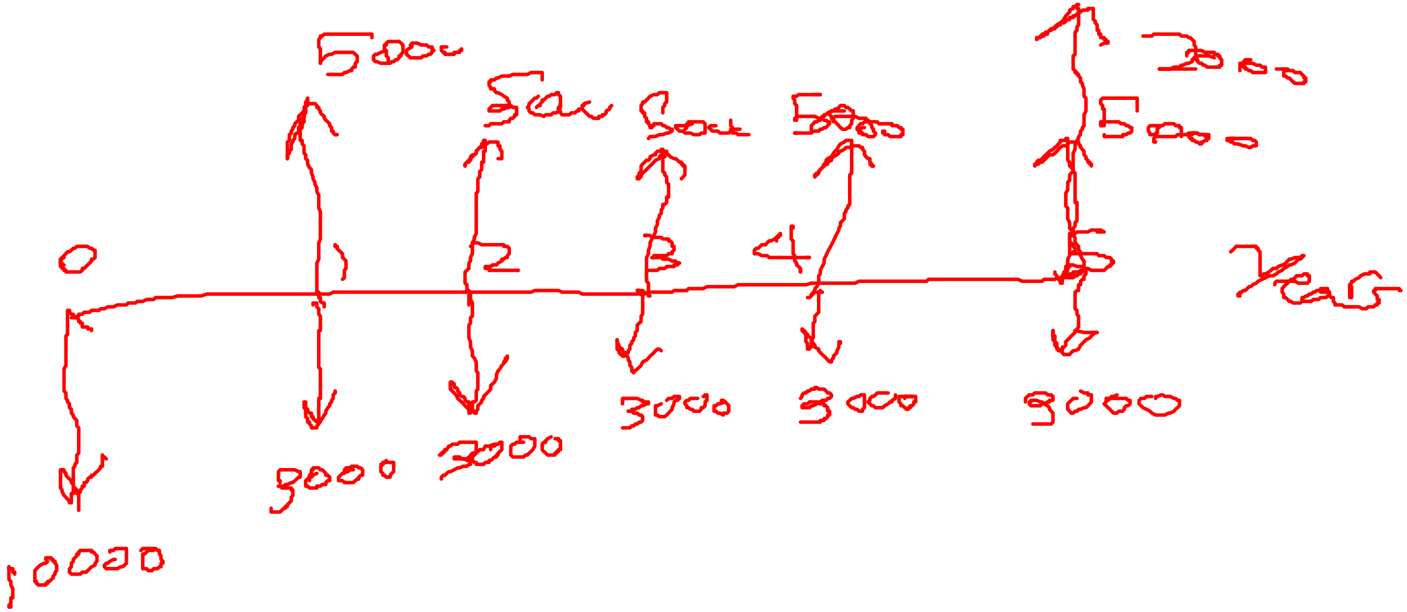
$I = 10000$

$R = 5000$

$n = 5$

$F = 3000$

$S = 2000$

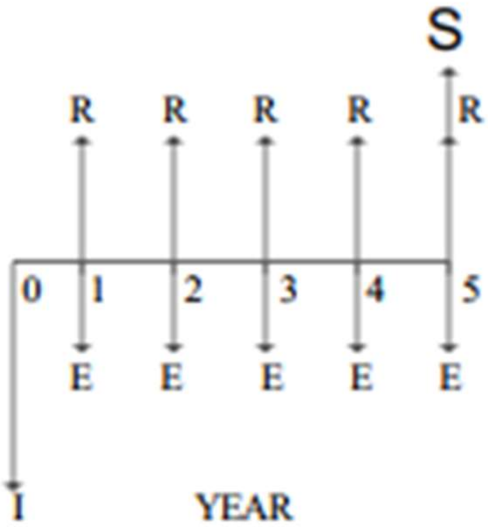


Example

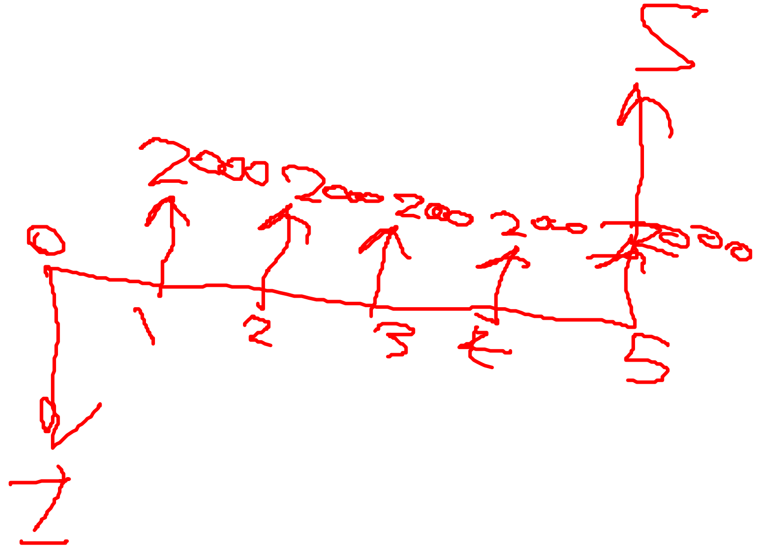
An investment of \$10,000 can be made that will produce uniform annual revenue of \$5,000 for 5 years and then have a salvage value of \$2,000 at the end of 5 years. Annual expenses will be \$3,000 at the end of each year for operating and maintaining the project.

I = 10,000
R = 5,000
E = 3,000
S = 2,000

NR = R − E
= 2000



CASH FLOW DIAGRAM

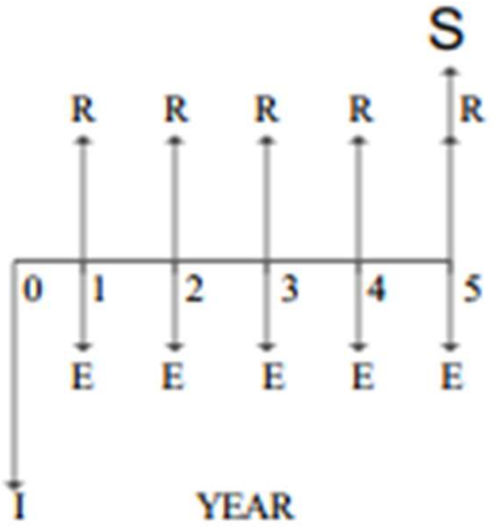


Example

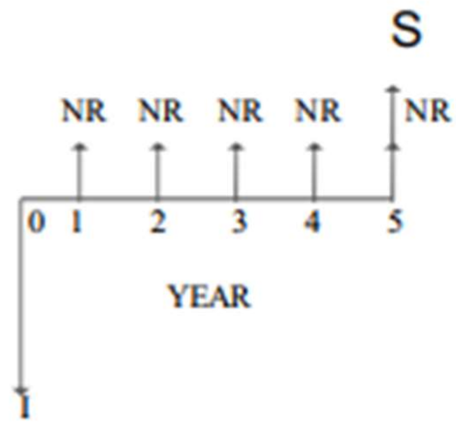
An investment of $10,000 can be made that will produce uniform annual revenue of $5,000 for 5 years and then have a salvage value of $2,000 at the end of 5 years. Annual expenses will be $3,000 at the end of each year for operating and maintaining the project.

I =10,000
R = 5,000
E = 3,000
S = 2,000

I =10,000
NR = R-E = 5,000 - 3,000 = 2,000
S= 2,000



CASH FLOW DIAGRAM



NET CASH FLOW DIAGRAM

Methods of Calculating Interest

- Simple interest: the practice of charging an interest rate only to an initial sum (principal amount).

- Compound interest: the practice of charging an interest rate to an initial sum and to any previously accumulated interest that has not been withdrawn.

الفائدة البسيطة: هي احتساب الفائدة على المبلغ الأصلي فقط.

الفائدة المركبة: هي احتساب الفائدة على المبلغ الأصلي بالإضافة إلى أي فوائد متراكمة لم تُسحب.

Simple interest

Example:

P = $1,000

i = 8%

N = 3 years

- Find: F

End of Year	Beginning Balance	Interest earned	Ending Balance
0 now			1000
1	1000	80	1080
2	1080	80	1160
3	1160	80	1240

Simple interest

Example:

P = $1,000

i = 8%

N = 3 years

- **Find: F**

End of Year	Beginning Balance	Interest earned	Ending Balance
0			$1,000
1	$1,000	$80	$1,080
2	$1,080	$80	$1,160
3	$1,160	$80	$1,240

Methods of Calculating Interest

Simple interest Formula

$F = P + (iP)N$

where

P = Principal amount

i = simple interest rate

N = number of interest periods

F = total amount accumulated at the end of period N

$F = P + (iP)N$

Simple interest

Example:

- P = \$1,000
- i = 8%
- N = 3 years
  - Find: F

$$\boxed{F = P + (iP)N}$$

End of Year	Beginning Balance	Interest earned	Ending Balance
0			\$1,000
1	\$1,000	\$80	\$1,080
2	\$1,080	\$80	\$1,160
3	\$1,160	\$80	\$1,240

$$F = 1000(1000 \times 0.08) \times 3 = \$1240$$

$$= 1000 + \left(\frac{8}{100} \times 1000\right) \times 3 = 1240$$

Example : Simple interest

You have agreed to loan a friend \$5000 for 5 years at a simple interest rate of 8% per year.
How much will your friend pay you at the end of 5 years?

$P = 5000$

$i = 8\%$

$n = 5$

$F = ?$

$F = P + (iP)N$

$= 5000 + (\frac{8}{100} * 5000) * 5$

$= \underline{7000}$ \$

Example : Simple interest

You have agreed to loan a friend \$5000 for 5 years at a simple interest rate of 8% per year.
How much will your friend pay you at the end of 5 years?

$$F = P + (iP)N$$

$$5000 + (\$5000)(0.08)(5\ \text{yr}) = \$7000$$

Methods of Calculating Interest

Compound interest: the practice of charging an interest rate to an initial sum and to any previously accumulated interest that has not been withdrawn

الفائدة المركبة :هي احتساب الفائدة على المبلغ الأصلي بالإضافة إلى أي فوائد متراكمة لم تُسحب.

Compound Interest

Example:

P = \$1,000

i = 8%

N = 3 years

- Find: F

End of Year	Beginning Balance	Interest earned	Ending Balance
0 now			1000
1	1000	80	1080
2	1080	86.4	1166.4
3	1166.4	93.31	1259.71

Compound Interest

Example:

- P = \$1,000
- i = 8%
- N = 3 years
  - Find: F

End of Year	Beginning Balance	Interest earned	Ending Balance
0			\$1,000
1	\$1,000	\$80	\$1,080
2	\$1,080	\$86.40	\$1,166.40
3	\$1,166.40	\$93.31	\$1,259.71

METHODS OF CALCULATING INTEREST

Compound interest

P = Principal amount

i = simple interest rate

N = number of interest periods

F = total amount accumulated at the end of period N

$$n = 0: \; P$$

$$n = 1: \; F_1 = P(1+i)$$

$$n = 2: \; F_2 = F_1(1+i) = P(1+i)^2$$

$$\boxed{N: \; F = P(1+i)^N}$$

In the formula $F = P(1+i)^N$, which is often used in economics and finance, the terms are defined as follows:

- F **(Future Value):** The value of an investment or a sum of money at a specified time in the future, considering the effect of interest over time. This represents the amount that will be received or accumulated after a certain period.
- P **(Present Value):** The current or initial amount of money being invested or lent. It is the principal sum of money before any interest has been applied.
- i **(Interest Rate per Period):** The rate at which the invested amount grows per period. This can be a percentage representing the growth factor due to interest. The rate is typically annual but can be applied to other periods such as monthly, quarterly, etc.
- N **(Number of Periods):** The total number of compounding periods during which the money will earn interest. For example, if interest is compounded annually, N would be the total number of years.

Compound Interest

Example:

P = \$1,000

i = 8%

N = 3 years

- **Find: F**

$$N : F = P(1+i)^N$$

End of Year	Beginning Balance	Interest earned	Ending Balance
0			\$1,000
1	\$1,000	\$80	\$1,080
2	\$1,080	\$86.40	\$1,166.40
3	\$1,166.40	\$93.31	\$1,259.71

$$F = \$1{,}000(1+0.08)^3$$
$$= \$1{,}259.71$$

$F = P(1+i)^n$

$F = 1000\left(1+\frac{8}{100}\right)^3$

Example :

You have agreed to loan a friend \$5000 for 5 years at a compound interest rate of 8% per year.
How much will your friend pay you at the end of 5 years?

Year	Total Principal (*P*) on Which Interest Is Calculated in Year *n*	Interest (*I*) Owed at the End of Year *n* from Year *n*'s Unpaid Total Principal	Total Amount Due at the End of Year *n*, New Total Principal for Year *n*+1
1	\$5000	\$5000 × 0.08 = 400	\$5000 + 400 = 5400
2	5400	5400 × 0.08 = 432	5400 + 432 = 5832
3	5832	5832 × 0.08 = 467	5832 + 467 = 6299
4	6299	6299 × 0.08 = 504	6299 + 504 = 6803
5	6803	6803 × 0.08 = 544	6803 + 544 = 7347

$$F = P(1+i)^n$$

$$= 5000\left(1+\frac{8}{100}\right)^5$$

$$= 7347$$

Example :

You have agreed to loan a friend \$5000 for 5 years at a compound interest rate of 8% per year.
How much will your friend pay you at the end of 5 years?

Year	Total Principal (*P*) on Which Interest Is Calculated in Year *n*	Interest (*I*) Owed at the End of Year *n* from Year *n*'s Unpaid Total Principal	Total Amount Due at the End of Year *n*, New Total Principal for Year *n*+1
1	\$5000	\$5000 × 0.08 = 400	\$5000 + 400 = 5400
2	5400	5400 × 0.08 = 432	5400 + 432 = 5832
3	5832	5832 × 0.08 = 467	5832 + 467 = 6299
4	6299	6299 × 0.08 = 504	6299 + 504 = 6803
5	6803	6803 × 0.08 = 544	6803 + 544 = 7347

$$\boxed{N : F = P(1+i)^N}$$

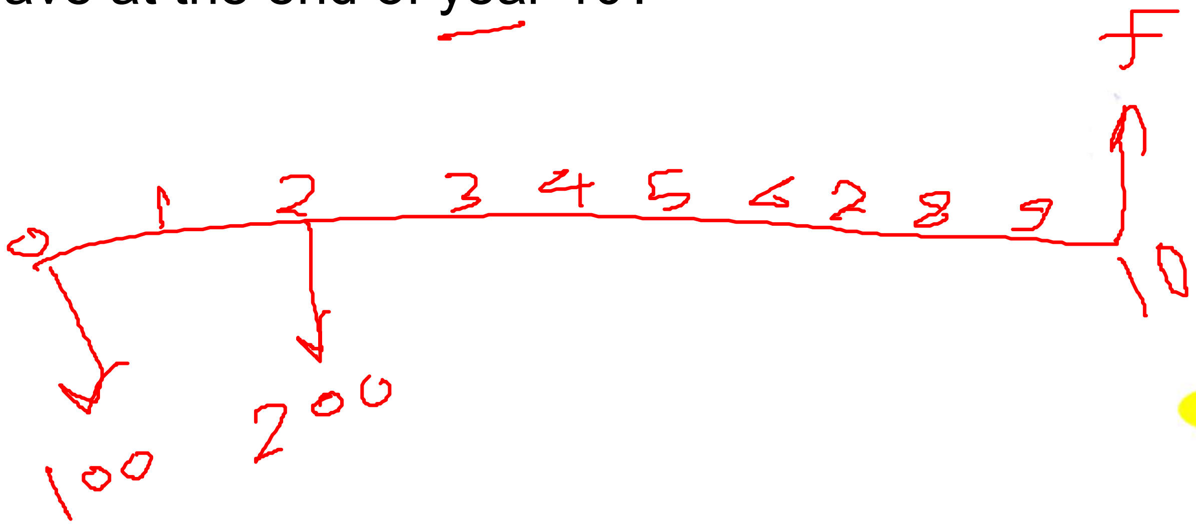
$$F = 5000(1 + 0.08)^5$$

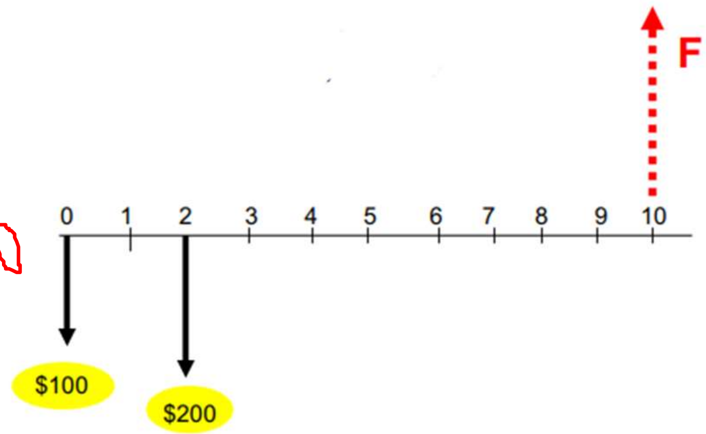
7347

PRACTICE PROBLEM 1

If you deposit $100 now (n=0) and $200 two years from now (n=2) in a savings account that pays 10% interest, how much would you have at the end of year 10?

$F = ?$
$P = 100$
$P_2 = 200$
$n = 10$
$i = 10\%$





PRACTICE PROBLEM 1

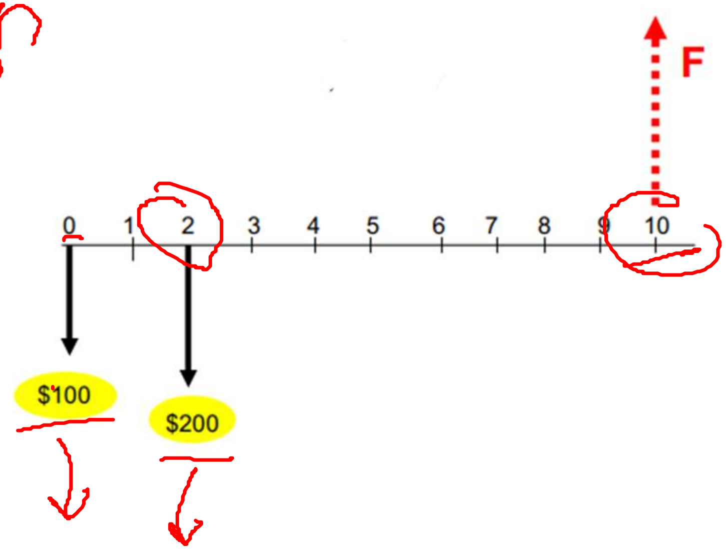
If you deposit $100 now (n=0) and $200 two years from now (n=2) in a savings account that pays 10% interest, how much would you have at the end of year 10?

$$F = P(1+i)^n$$

$$F_1 = 100\left(1+\frac{10}{100}\right)^{10} = 259$$

$$F_2 = 200\left(1+\frac{10}{100}\right)^{8} = 429$$

$$F = F_1 + F_2 = 628\cancel{8}$$



PRACTICE PROBLEM 1

If you deposit \$100 now (n=0) and \$200 two years from now (n=2) in a savings account that pays 10% interest, how much would you have at the end of year 10?

$$\boxed{N: F = P(1+i)^N}$$

$$F = F1 + F2$$

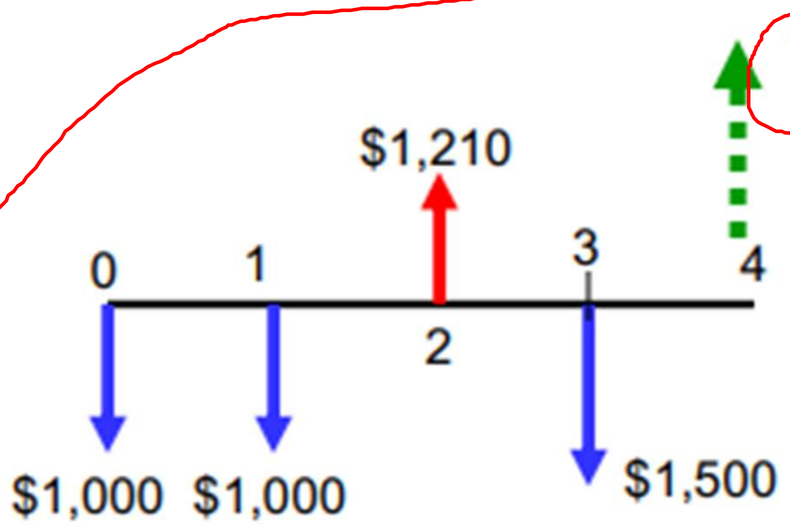
$$F1 = 100(1 + 0.10)^{10} = \$259$$

$$F2 = 200(1 + 0.10)^{8} = \$429$$

$$F = 259 + 429 = \$688$$

PRACTICE PROBLEM 2

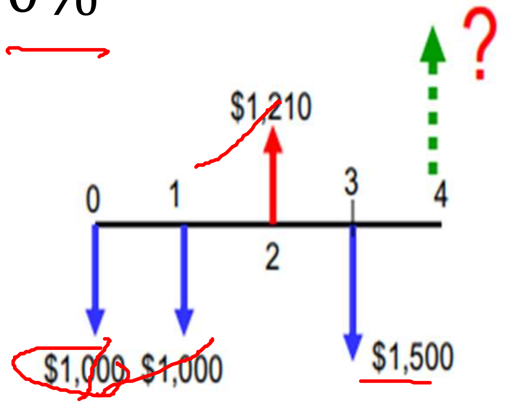
Consider the following sequence of deposits and withdrawals over a period of 4 years. If you earn 10% interest, what would be the balance at the end of 4 years?



PRACTICE PROBLEM 2

$i = 10\%$

End of Period	Beginning balance	Deposit made	Withdraw	Ending balance
n=0	0	1000	—	1000
n=1	1100	1000	—	2100
n=2	2310	—	1210	1100
n=3	1210	1500	—	2710
n=4	2980	—	—	2980

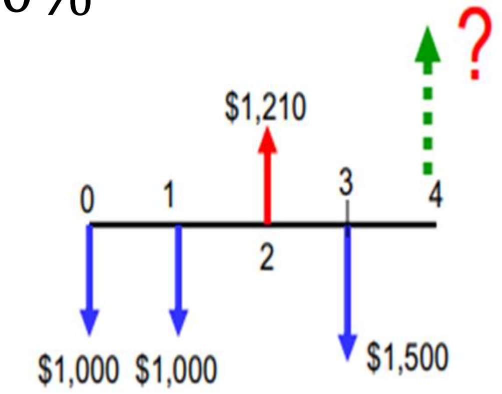


$F = P(1+i)^n$

PRACTICE PROBLEM 2

$i = 10\%$

End of Period	Beginning balance	Deposit made	Withdraw	Ending balance
n=0	0	1000	0	1000
n=1	1100	1000	0	2100
n=2	2310	0	1210	1100
n=3	1210	1500	0	2710
n=4	2981	0	0	?



$$F = P(1+i)^n$$

$F1 = 1000(1+0.10)^1 = \$1100$

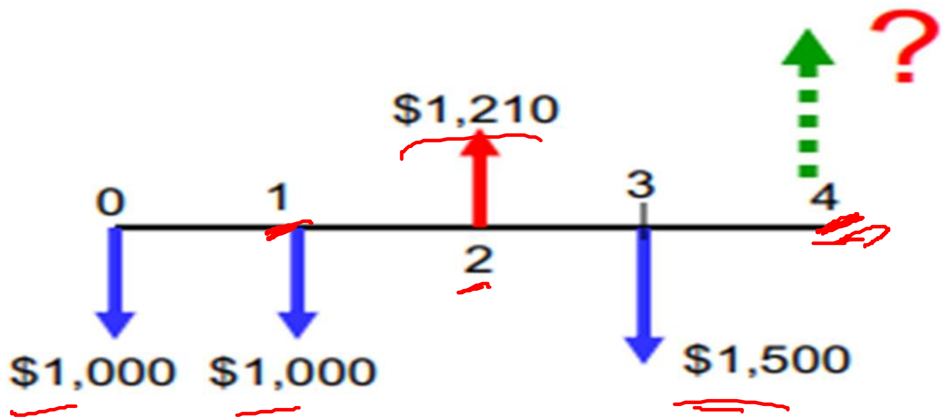
$F2 = 2100(1+0.10)^1 = \$2310$

$F3 = 1100(1+0.10)^1 = \$1210$

$F4 = 2710(1+0.10)^1 = \$2981$

PRACTICE PROBLEM 2

$i = 10\%$



$F = P(1+i)^n$

$F_1 = 1000(1+0.1)^4 =$

$F_2 = 1000(1+0.1)^3 =$

$F_3 = 1210(1+0.1)^2 =$

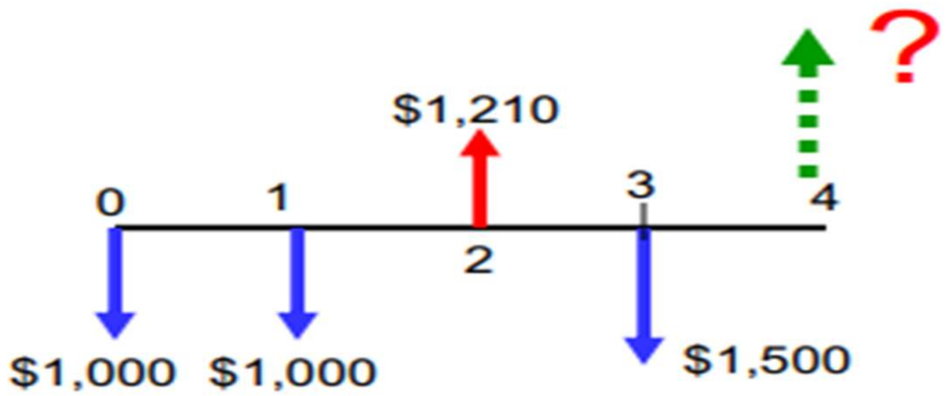
$F_4 = 1500(1+0.1) =$

$F = F_1 + F_2 + F_3 + F_4$

$F = 2981$ #

PRACTICE PROBLEM 2

$i = 10\%$



$$F = (\text{F/P},i,n) = P(1+i)^n$$

$$F1 = (\text{F/P},0.1,4) = 1000(1+0.10)^4$$

$$F2 = (\text{F/P},0.1,3) = 1000(1+0.10)^3$$

$$F3 = (\text{F/P},0.1,2) = 1210(1+0.10)^2$$

$$F4 = (\text{F/P},0.1,1) = 1500(1+0.10)^1$$

$$F = F1 + F2 - F3 + F4 \quad = 2{,}981$$

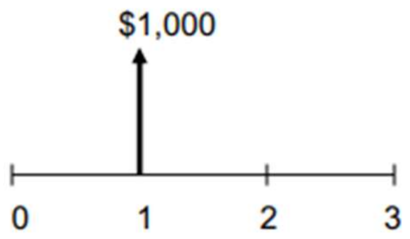
CHAPTER 2

TIME VALUE OF MONEY

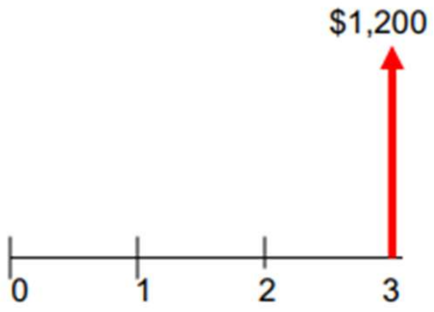
Economic Equivalence

Discussion

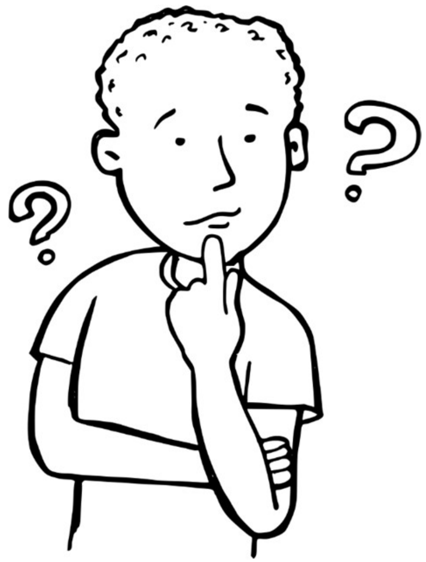
Consider the following two cash receipts.
Which option would you prefer?



Option 1

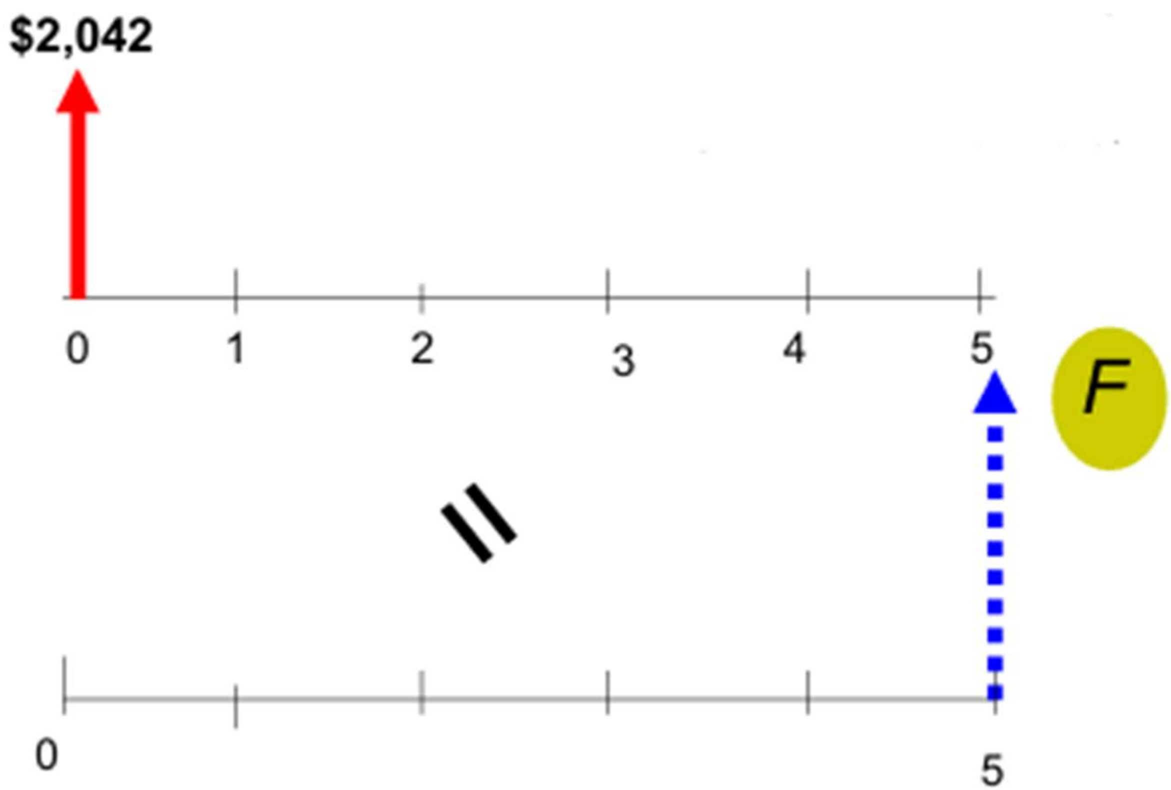


Option 2



Practice Problem 1

If you deposit $2,042 today in a savings account that pays 8% interest annually, how much would you have at the end of 5 years?



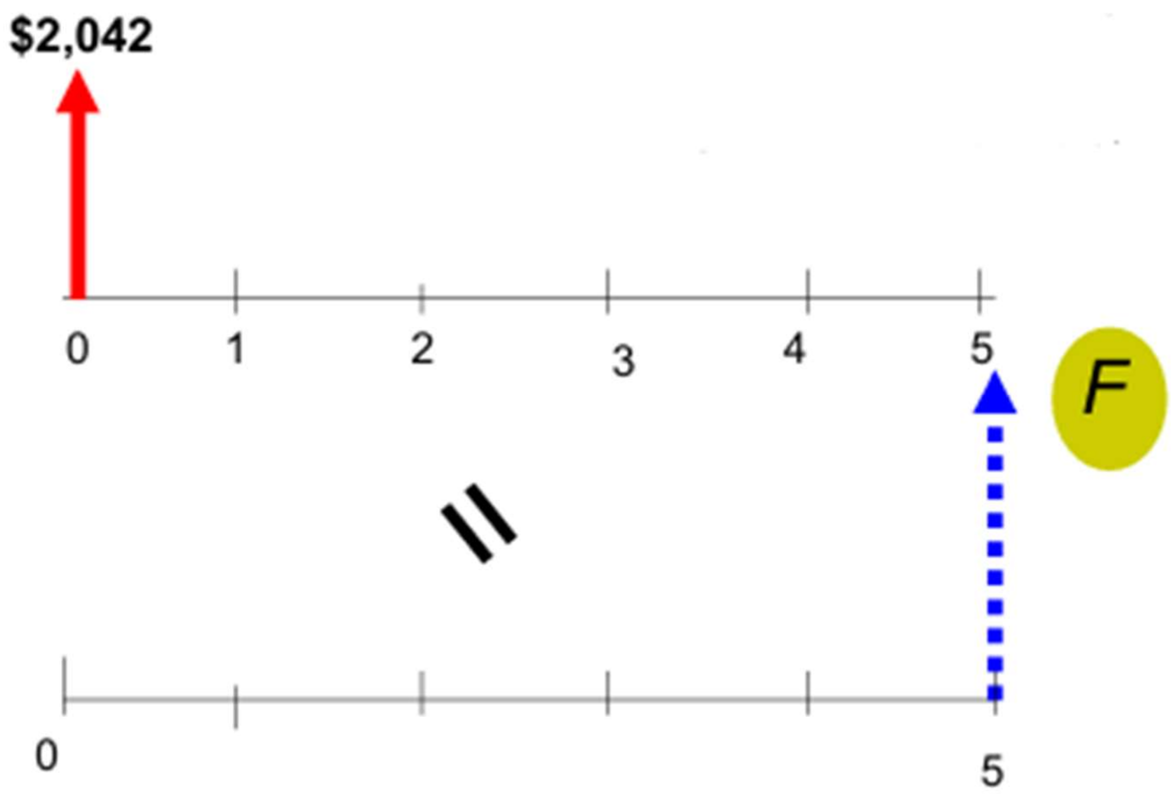
$$F = P(1+i)^n$$

$$F = 2042\left(1+\frac{8}{100}\right)^5$$

$$F = 3000\ \$$$

Practice Problem 1

If you deposit $2,042 today in a savings account that pays 8% interest annually, how much would you have at the end of 5 years?



$$F = P(1+i)^n$$

$$F = 2{,}042\$(1+0.08)^5$$

$$F = \$3000$$

Example 2: Finding the Present Value

Problem:

You will receive $1,200 three years from now, and the annual interest rate is 5%. What is the equivalent value of that money today?

i

$$F = P(1+i)^n$$

$$P = F(1+i)^{-n}$$

$$= 1200 \times (1+\frac{5}{100})^{-3} = 1036\$$$

Example 2: Finding the Present Value

Problem:

You will receive $1,200 three years from now, and the annual interest rate is 5%. What is the equivalent value of that money today?

Solution:

To find the present value (PV), we use the formula:

$$PV = \frac{F}{(1+i)^N}$$

Plugging in the values:

$$PV = \frac{1,200}{(1+0.05)^3} = \frac{1,200}{1.157625} \approx 1,036.68$$

Answer: The equivalent value of $1,200 three years from now is approximately $1,036.68 today.

Example 3: Comparing Two Cash Flows

Problem:

You have two investment options:

1. Receive $500 today.
2. Receive $550 one year from now.

   If the annual interest rate is 8%, which option is more valuable?

Solution:

To compare these two options, we need to find the present value of the future amount ($550) and compare it with $500 today.

To compare these two options, we need to find the present value of the future amount ($550) and compare it with $500 today.

$n \xrightarrow{+} F$

$P \xleftarrow{-} F$

$i = 8\%$

$n = 1$

$$P = F(1+i)^{-n}$$

$$P = 550\left(1 + \frac{8}{100}\right)^{-1}$$

op2 $\boxed{P = 509 \#}$

op1 500

To compare these two options, we need to find the present value of the future amount ($550) and compare it with $500 today.

Using the formula for present value:

$$PV = \frac{F}{(1+i)^N}$$

Plugging in the values:

$$PV = \frac{550}{(1+0.08)^1} = \frac{550}{1.08} \approx 509.26$$

Comparison:

- Option 1: $500 today
- Option 2: Equivalent to $509.26 today

Answer: Option 2 is more valuable since its present value ($509.26) is higher than $500.

Alternatively

$$FV = P(1+i)^N$$

We can find FV of 500 $ you have today.

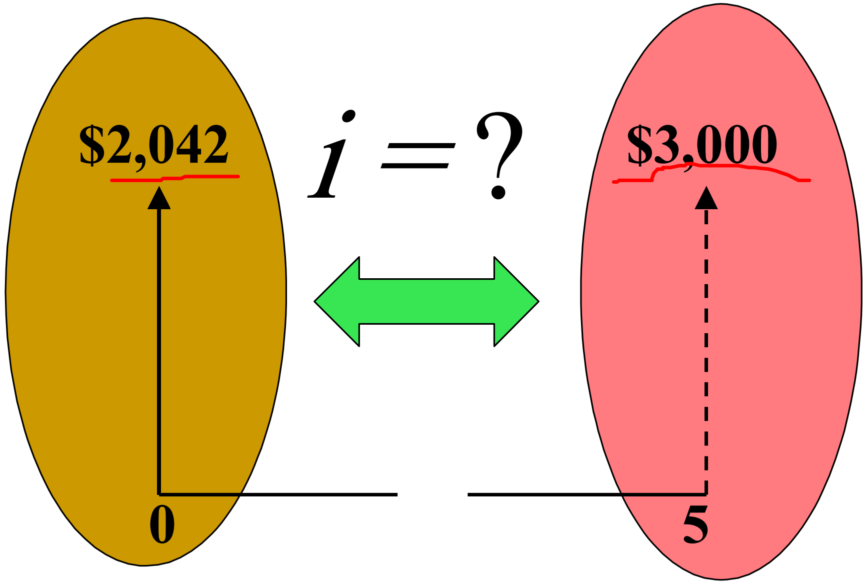
So,

$FV = 500(1+0.08)^1$
$FV = 540$

Since Option 1 has lower FV as compared to 550, so we will go for Option 2.

Example 2.2

At what interest rate
would these two amounts be equivalent?



Equivalence Between Two Cash Flows

$$F = P(1+i)^n$$

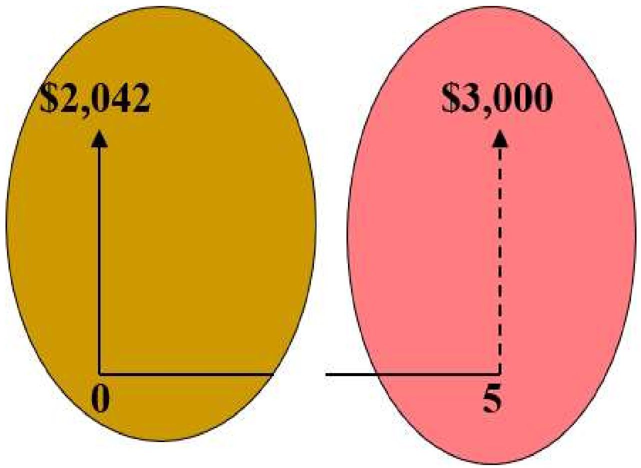
$$3000 = \cancel{2042}(1+i)^5 \quad \div 2042$$

$$1.47 = (1+i)^5 \quad \sqrt[5]{}$$

$$i+1 = (1.47)^{1/5}$$

$$i = (1.47)^{1/5} - 1$$

$$i = 0.08 = 8\%$$



Equivalence Between Two Cash Flows

$$F = P(1+i)^n$$

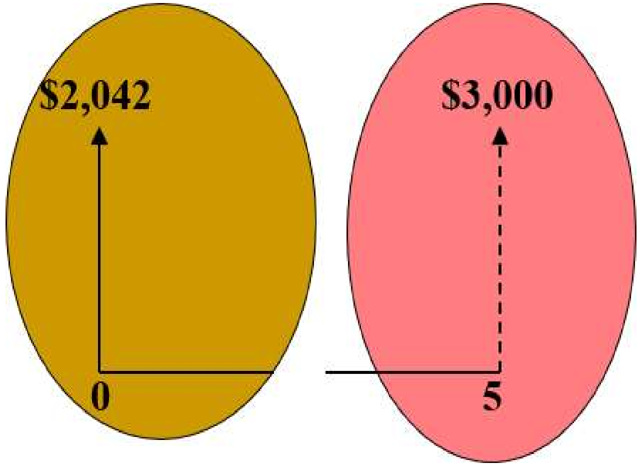
$$3{,}000\$ = 2{,}042\$(1+i)^5$$

$$3{,}000\$/2{,}042\$ = (1+i)^5$$

$$(1.47)^{1/5} = 1 + i$$

$$i = (1.47)^{\frac{1}{5}} - 1$$

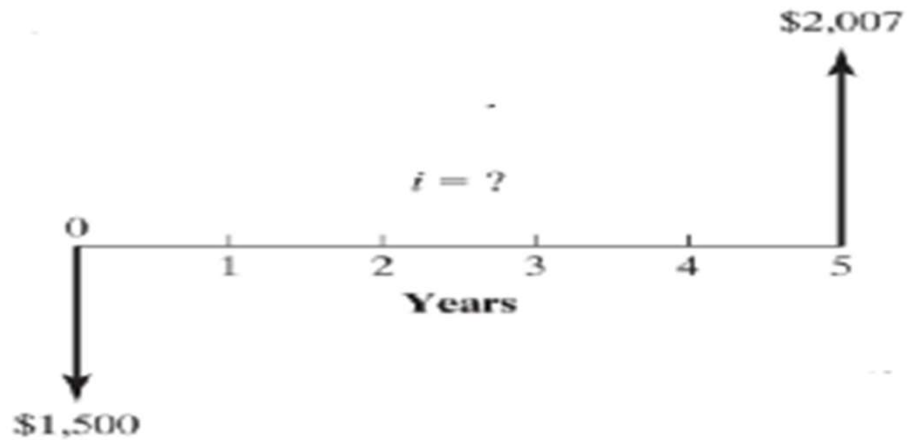
$$i = 0.079 = 8\%$$



Example

At what interest rate would these two payments be equivalent?

Given: F = \$2,007, N = 5 years, P = \$1,500, Find: i



Example

At what interest rate would these two payments be equivalent?

Given: F = $2,007, N = 5 years, P = $1,500, Find: i

$$2{,}007\$/1{,}500\$ = (1+i)^5$$

$$F = P(1+i)^n$$

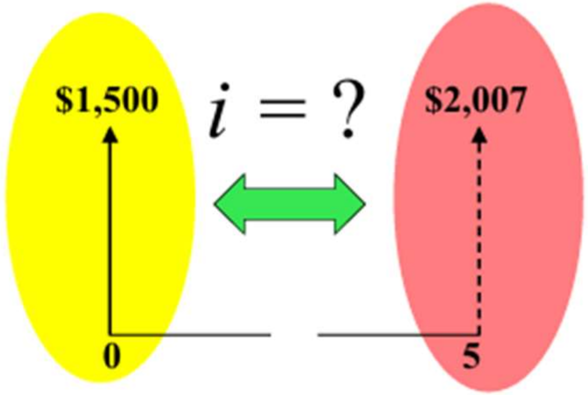
$$2{,}007\$ = 1{,}500\$(1+i)^5$$

$$2{,}007\$/1{,}500\$ = (1+i)^5$$

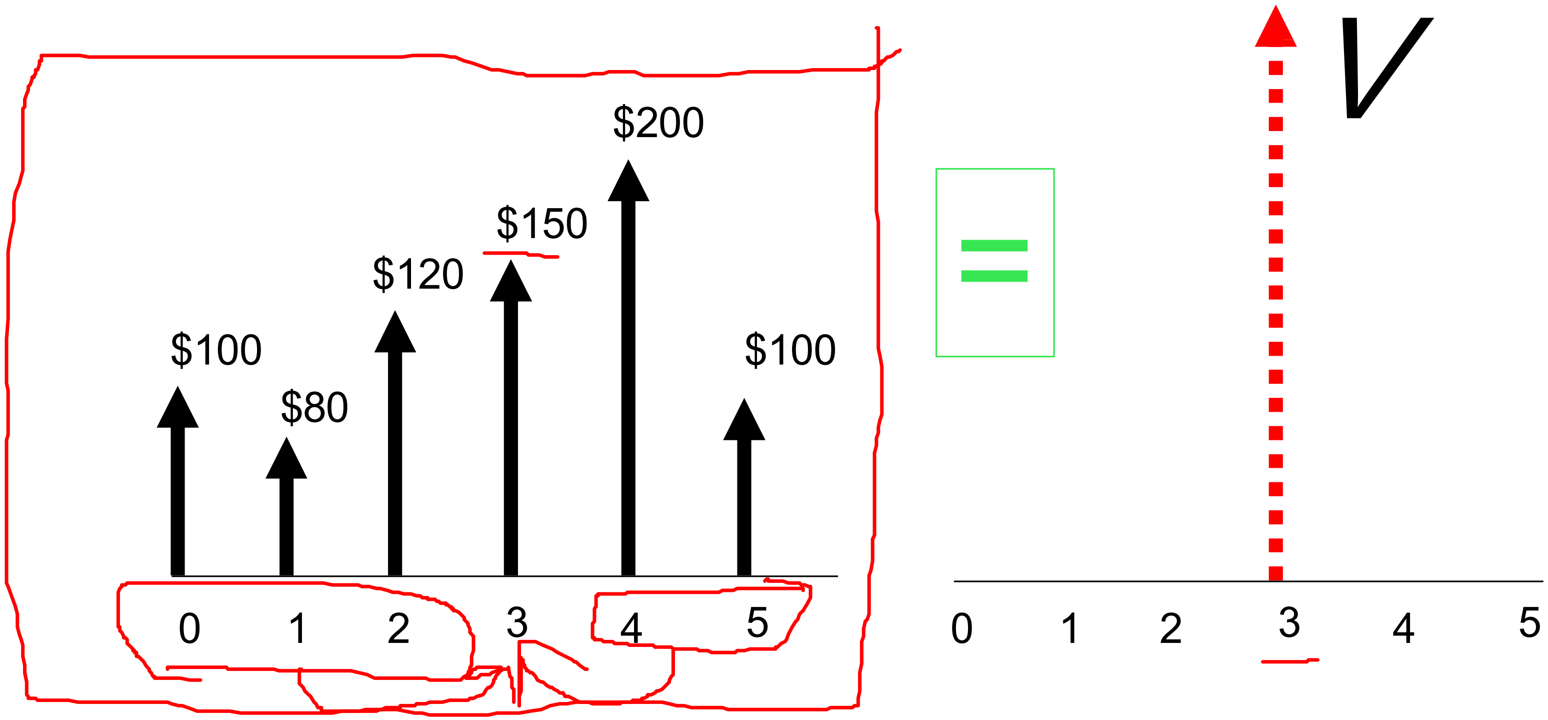
$$(2{,}007\$/1{,}500\$)^{1/5} = 1+i$$

$$i = (2{,}007\$/1{,}500\$)^{1/5} - 1$$

$$i = 0.06 = 6\%$$



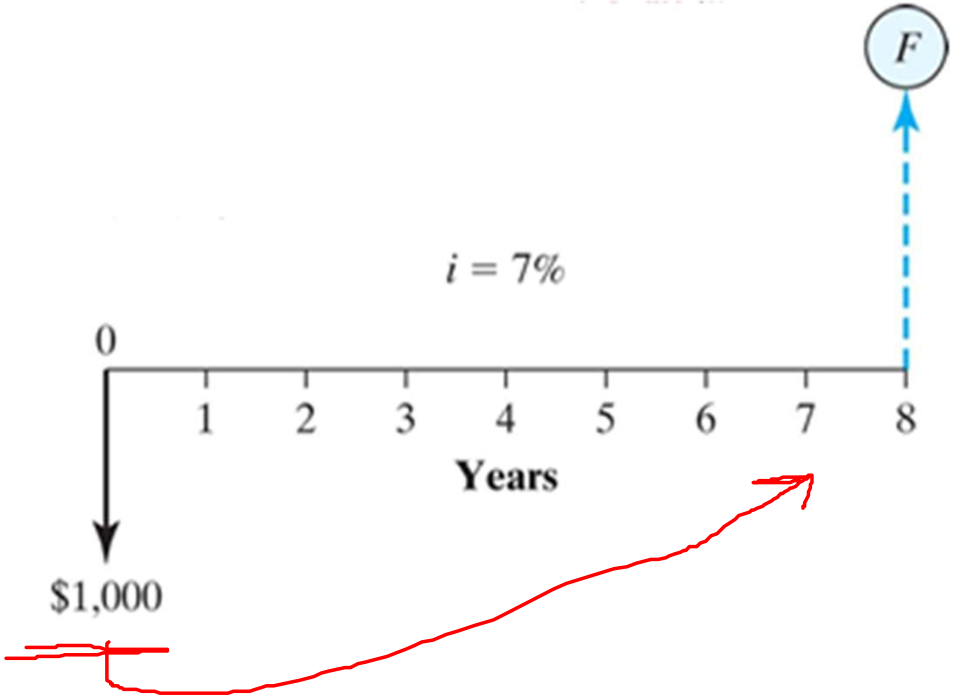
Example 2.3



Compute the equivalent lump-sum amount at $n = 3$ at 10% annual interest.

Example

If you had $1,000 now and invested it at 7% interest compounded annually, how much would it be worth in 8 years?



Example

If you had $1,000 now and invested it at 7% interest compounded annually, how much would it be worth in 8 years?

$$F = P(1+i)^n$$

$$F = 1,000\$(1+0.07)^8$$

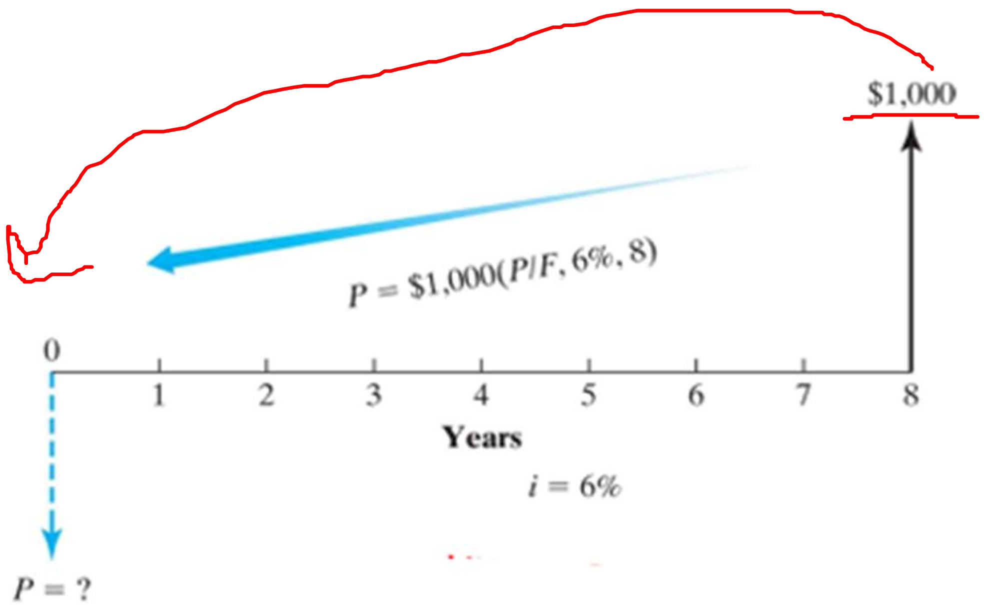
$$F = 1,000\$(1+0.07)^8$$

$$F = 1,718.19\$$$

Given :P = $1,000, i = 7%, and N = 8 years; Find : F

Example

Given: F = $1000, i= 6%, and N = 8 years



Example

Given: F = $1000, i= 6%, and N = 8 years

$$P = F(1+i)^{-n}$$

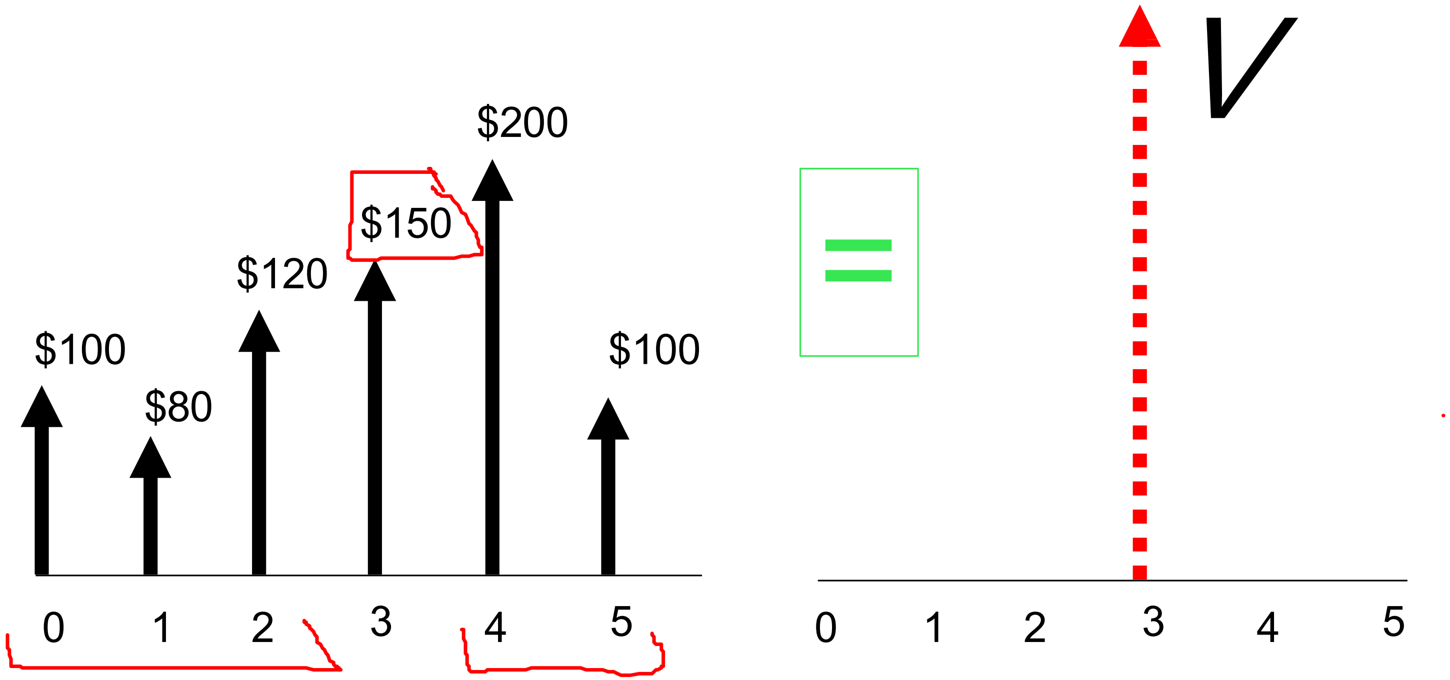
$$F = P(1+i)^n$$

$$P = F(1+i)^{-n}$$

$$P = 1{,}000\$(1+0.06)^{-8}$$

$$P = 627.4\ \$$$

Example 2.3



Compute the equivalent lump-sum amount at $n = 3$ at 10% annual interest.

Example 2.3

$F = P(1+i)^n$

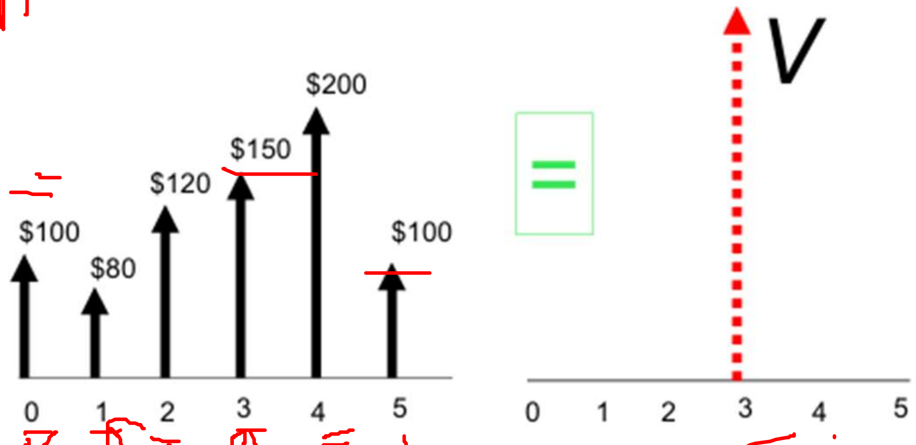
Step 1 (from zero to 3)

$100(1+\frac{10}{100})^3 + 80(1+0.1)^2 + 120(\ )^1 =$

$P = F(1+i)^{-n}$

Step 2 (from 6 to 3)

$200(1+\frac{10}{100})^{-1} + 100(\ )^{-2} =$



i= 10%.

Step 3 (Equivalent worth calculation at n = 3)

$361.9 + 264.6 + 150 = 776.36$ \$

Example 2.3

The cash flows given in figure , FIND:V3(equivalent worth at n = 3) and i= 10%.

Step 1 (from zero to 3)

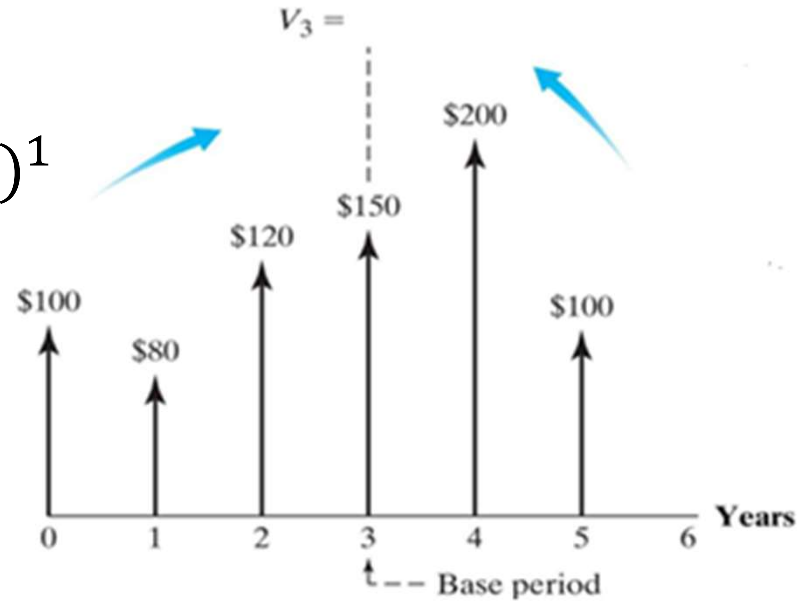
$$100\$(1+0.1)^3 + 80\$(1+0.1)^2 + 120\$(1+0.1)^1 = 361.9\$$$

Step 2 (from 6 to 3)

$$200\$(1+0.1)^{-1} + 100\$(1+0.1)^{-2} = 264.46\$$$

Step 3 (Equivalent worth calculation at n = 3)

$$V3 = 361.9\$ + 264.46\$ + 150\$ = 776.36\$$$



$$F = P(1+i)^n$$

Calculate the equivalent value for each cash flow:

- At $n = 0$:

$$V_0 = 100 \times (1 + 0.10)^3 = 100 \times 1.331 = 133.1$$

- At $n = 1$:

$$V_1 = 80 \times (1 + 0.10)^2 = 80 \times 1.21 = 96.8$$

- At $n = 2$:

$$V_2 = 120 \times (1 + 0.10)^1 = 120 \times 1.1 = 132$$

- At $n = 3$:

$$V_3 = 150 \times (1 + 0.10)^0 = 150 \times 1 = 150$$

- At $n = 4$:

$$V_4 = 200 \times (1 + 0.10)^{-1} = 200 \times \frac{1}{1.1} \approx 181.82$$

- At $n = 5$:

$$V_5 = 100 \times (1 + 0.10)^{-2} = 100 \times \frac{1}{1.21} \approx 82.64$$

. Sum the equivalent values at $n = 3$:

$$V = V_0 + V_1 + V_2 + V_3 + V_4 + V_5$$

$$V = 133.1 + 96.8 + 132 + 150 + 181.82 + 82.64 = 776.36$$

When calculating the equivalent lump-sum amount at n=3, we need to bring all cash flows to their equivalent values at n=3. For cash flows that occur after n=3 (i.e., at n=4 and n=5), we need to **discount** them back to n=3. This is why negative sign is used with the exponents in the formula to calculate their present value at n=3.

Practice Problem

Given: P = \$3,000, F = \$6,000 and i= 12%, Find: N

$$F = P(1+i)^n$$

$$6000 = 3000(1+0.12)^n$$

$$2 = (1.12)^n \quad \log$$

$$\log 2 = n \log(1.12)$$

$$n = \frac{\log 2}{\log 1.12} = 6.12 \text{ years}$$

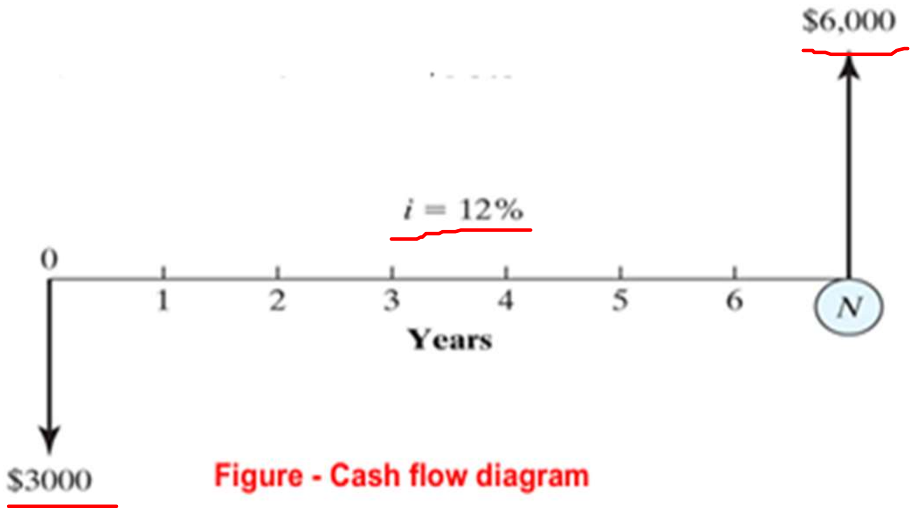


Figure - Cash flow diagram

Practice Problem

Given: P = $3,000, F = $6,000 and i= 12%, Find: N

$$F = P(1+i)^n$$

$$6000\$ = 3000\$(1+0.12)^n$$

$$(6000\$ /3000) = (1.12)^n$$

$$log(2) = nlog(1.12)$$

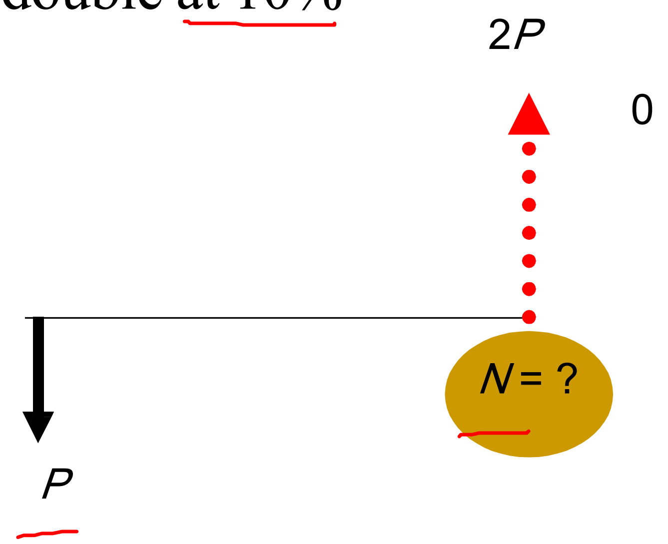
$$log(2)/log(1.12) = n$$

$$n = 6.116 = 6.12\ years$$

$$F = P(1+i)^n$$

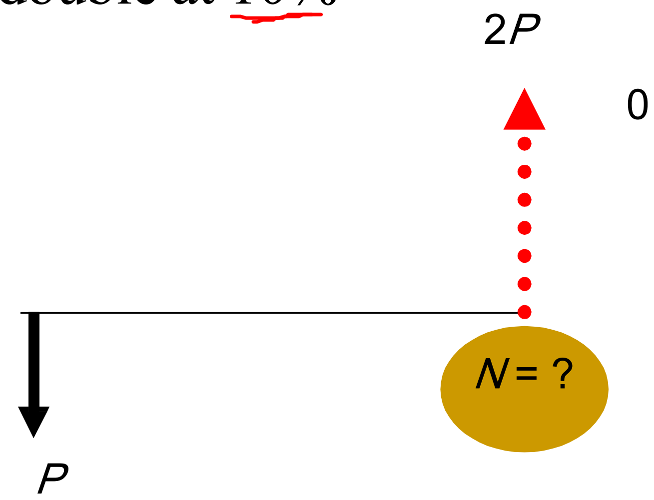
Practice Problem

- How many years would it take an investment to double at 10% annual interest?



Practice Problem

- How many years would it take an investment to double at 10% annual interest?



$$F = 2P = P(1+0.10)^N$$

$$2 = 1.1^N$$

$$\log 2 = N \log 1.1$$

$$N = \frac{\log 2}{\log 1.1}$$

$$= 7.27 \text{ years}$$

Log Natural

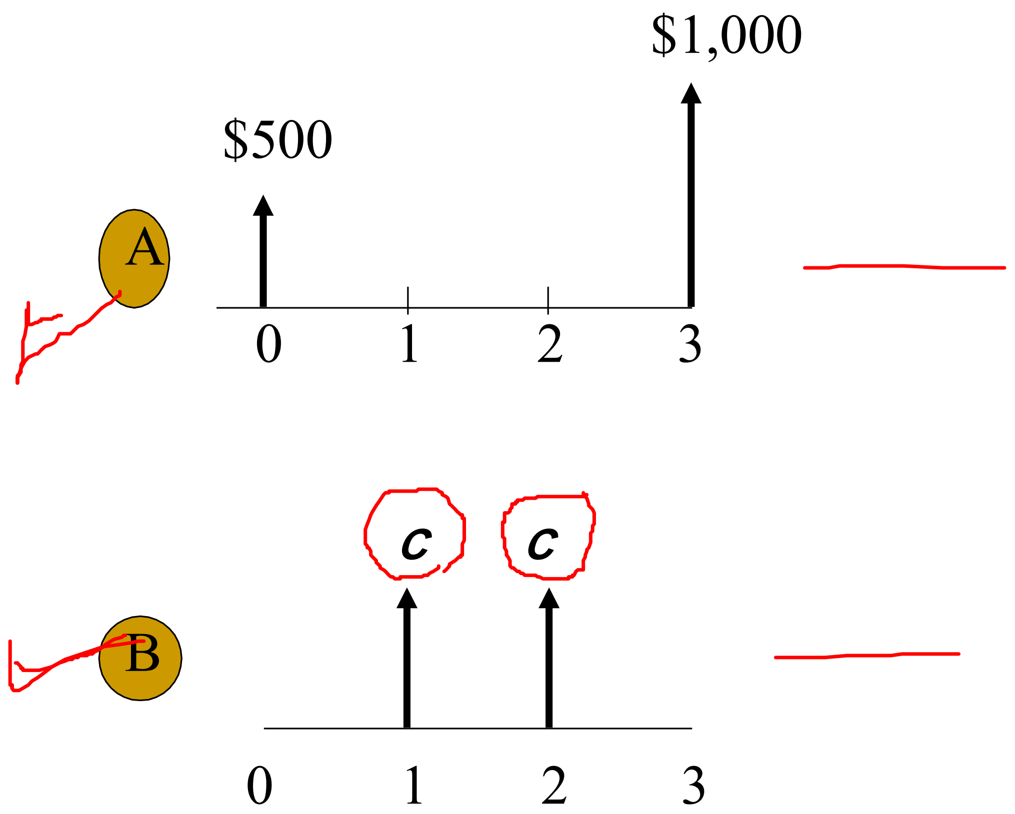
CHAPTER 2

TIME VALUE OF MONEY

Economic Equivalence

Practice Problem

Given: $i = 10\%$, Find: C that makes the two cash flow streams to be indifferent or equal.



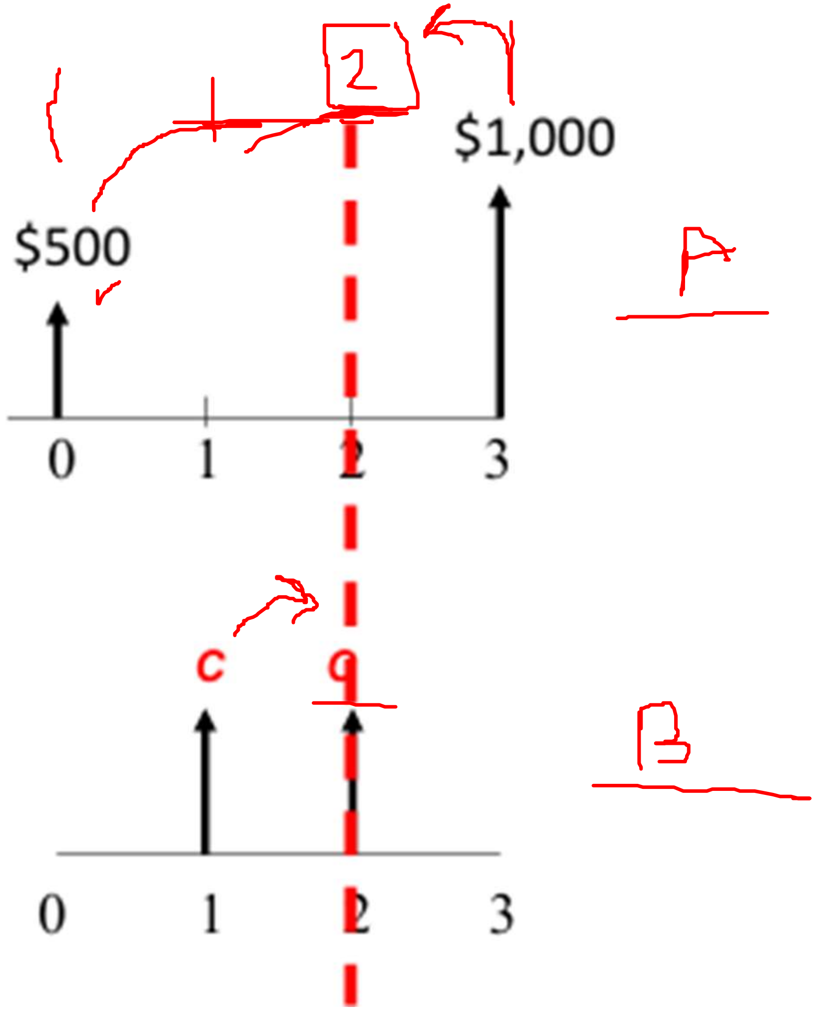
Practice Problem

Given: i= 10%, Find: C that makes the two cash flow streams to be indifferent

Step 1: Select the base period to use, say n= 2

Step 2: Find the equivalent lump sum value at n= 2 for both A and B.

Step 3: Equate both equivalent values and solve for unknown C.



Practice Problem

Given: i= 10%, Find: C that makes the two cash flow streams to be indifferent

$F = P(1+i)^n$

$P = F(1+i)^{-n}$

For A

$V_2 = 500(1+0.10)^2 + 1000(1+0.10)^{-1}$

$= 1514\ \$$

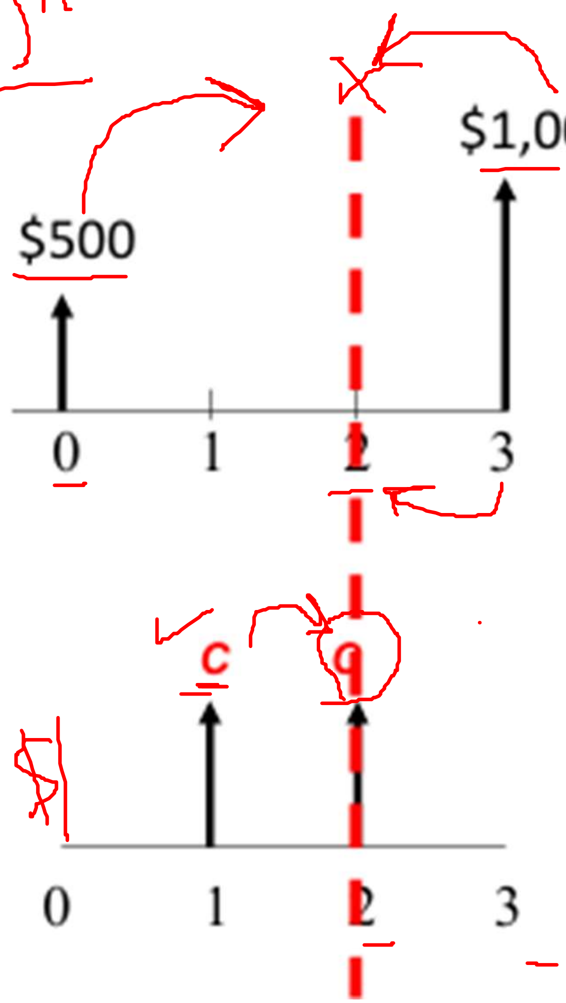
For B

$V_2 = C(1+0.10)^1 + C = 2.1C$

$(V_2)_A = (V_2)_B$

$2.1C = 1514$

$C = 721\ \$$



Practice Problem

Given: i= 10%, Find: C that makes the two cash flow streams to be indifferent

For A :

$$V2 = 500\$(1+0.1)^2 + 1{,}000\$(1+0.1)^{-1}$$
$$= 1{,}514\$$$

For B :

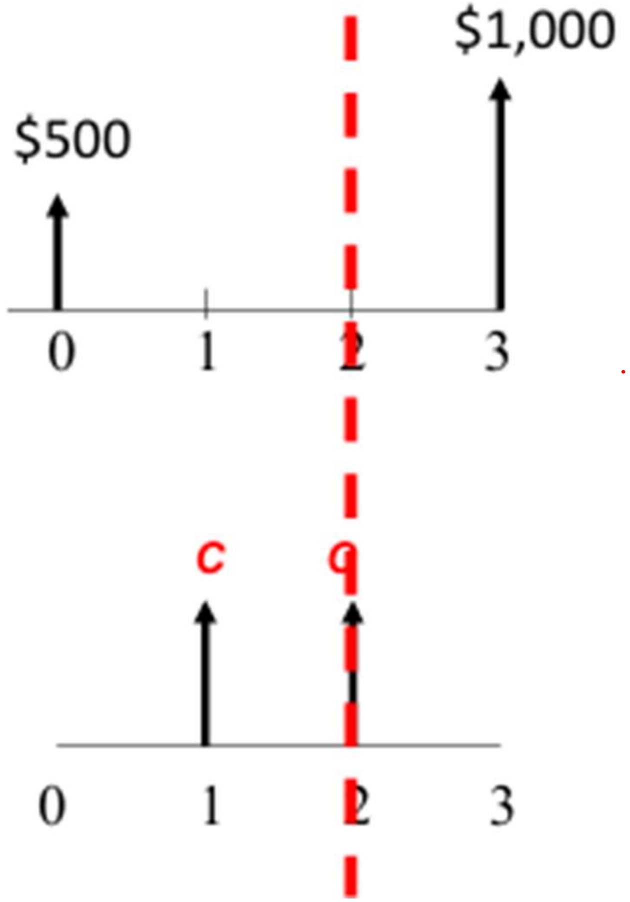
$$V2 = C(1+0.1)^1 + C$$
$$= 2.1C$$

To Find C :

$$(V2)B = (V2)A$$

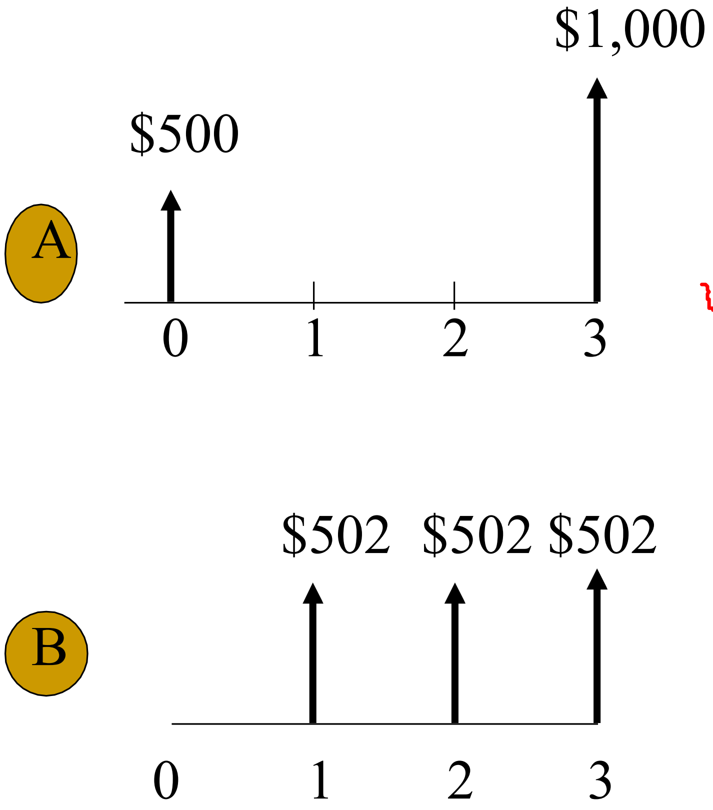
$$2.1C = 1{,}514\$$$

$$C = 721\$$$



Practice Problem

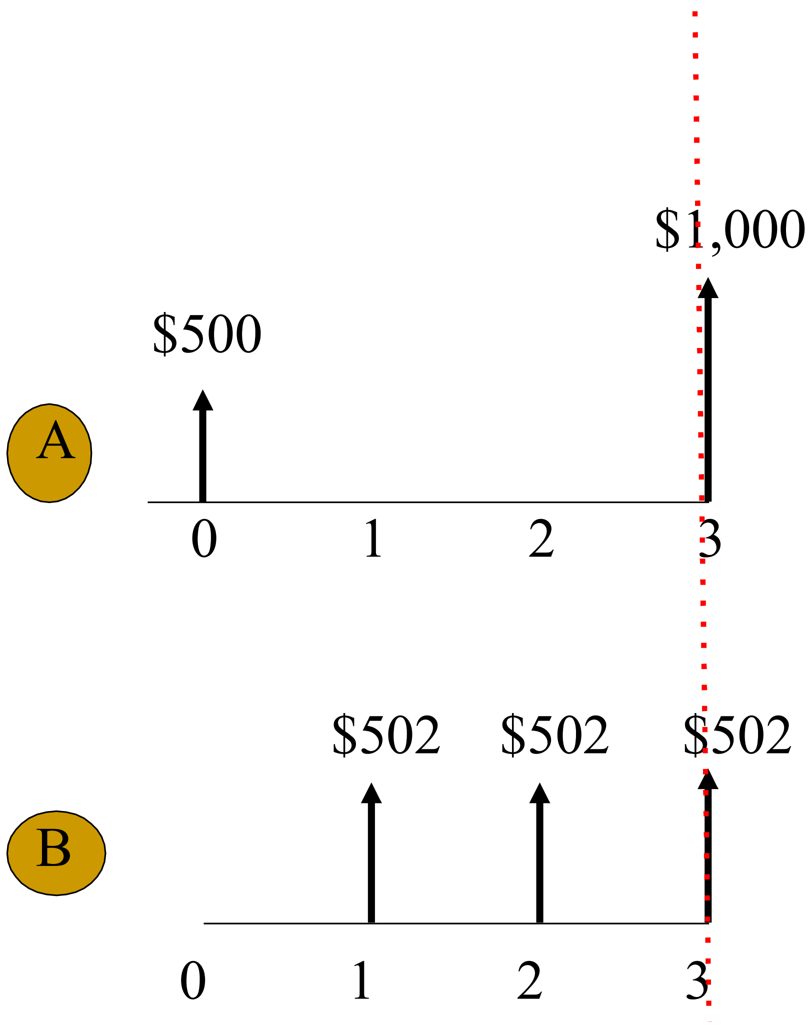
At what interest rate would you be indifferent between the two cash flows?



It simply means: **the interest rate at which both cash flow options are worth the same.**

Solution

- Step 1: Select the base period to compute the equivalent value (say, $n = 3$)
- Step 2: Find the net worth of each at $n = 3$.



Solution

For A $V_3 = 500(1+i)^3 + 1000$

For B $V_3 = 502(1+i)^2 + 502(1+i)^1 + 502$

iteration method

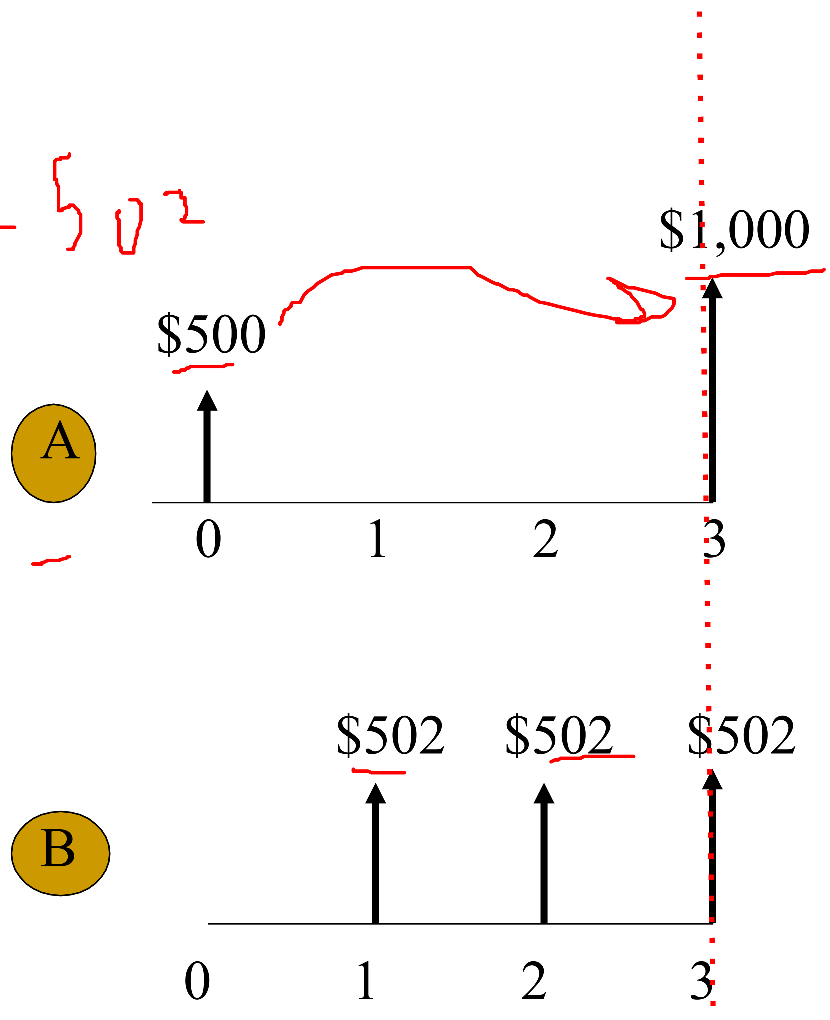
$i = 9 \quad 2$

$i = 10 \quad 4 \quad -$

$i = 9\%$

$(V_3)_A = 1630$

$(V_3)_B = 1630$



Solution

At what interest rate would you be indifferent between the two cash flows?

For A :

$$F3 = 500\$(1+i)^3 + 1{,}000\$$$

For B :

$$F3 = 502\$(1+i)^2 + 502\$(1+i)^1 + 502$$

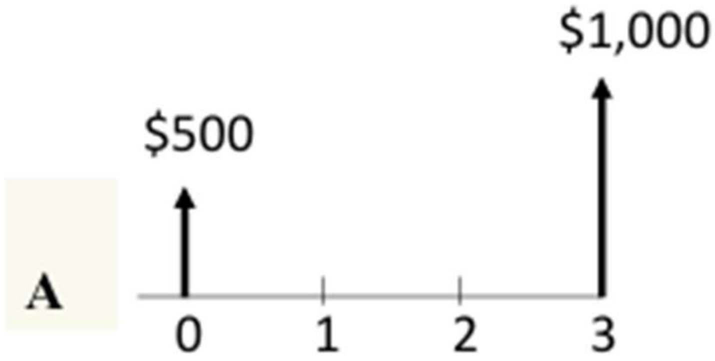
To Find i (use trial and error) :

Assume $i = 8\%$ *and calculate* $F3$ *at each case*

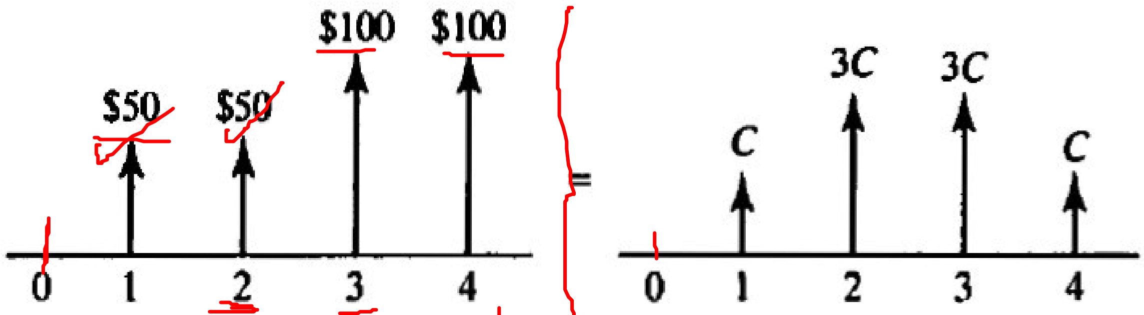
$$(F3)A = 1{,}630$$

$$(F3)B = 1{,}630$$

$$\boxed{i = 8}$$ %



2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:



$P = F($

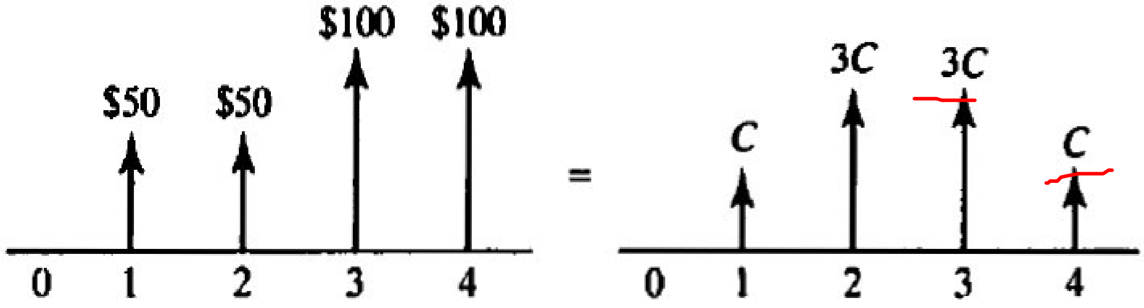
$P = F(1+i)^{-n}$

$P = 50(1+0.12)^{-1} +$
$50(\quad)^{-2} +$
$100(\quad)^{-3} +$
$100(\quad)^{-4}$

$= 219.032$

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:

$$P = F(1+i)^{-n}$$



For A :

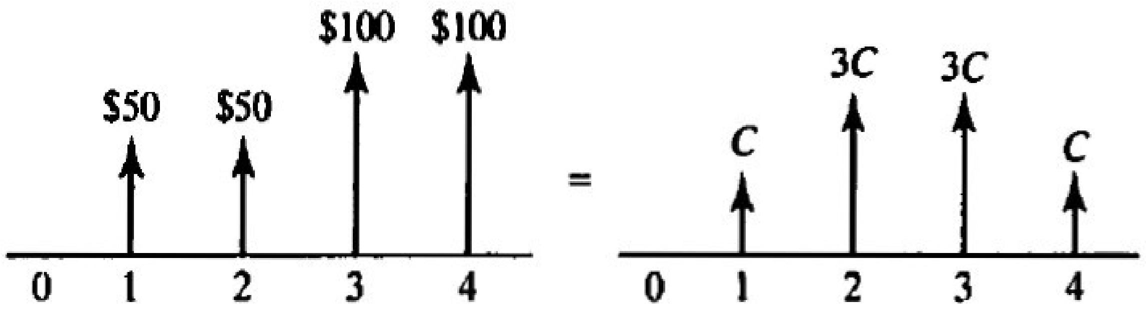
$$(P)A = 50\$(1+0.12)^{-1} + 50\$(1+0.12)^{-2} + 100\$(1+0.12)^{-3} + 100\$(1+0.12)^{-4} = 219.232\$$$

For B :

$$(P)_A = C(1.12)^{-1} + 3C(\;)^{-2} + 3C(\;)^{-3} + C(\;)^{-4}$$

$$= 6.055\,C$$

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:



For A :

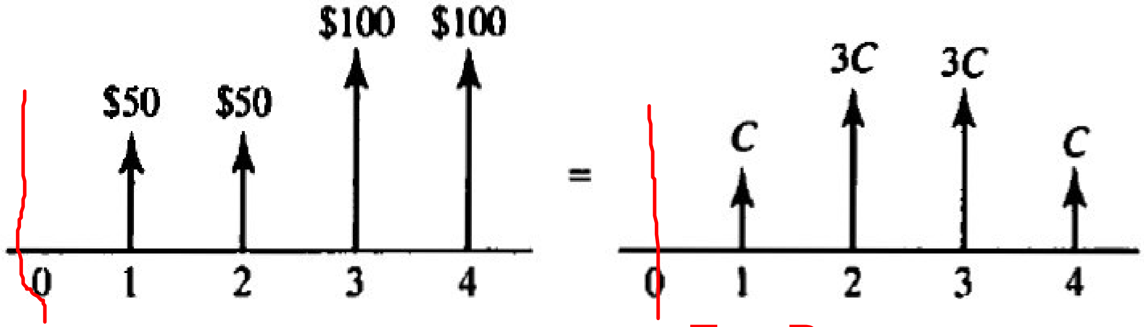
$$(P)A = 50\$(1+0.12)^{-1} + 50\$(1+0.12)^{-2} + 100\$(1+0.12)^{-3} + 100\$(1+0.12)^{-4} = 219.232\$$$

For B :

$$(P)B = C(1+0.12)^{-1} + 3C(1+0.12)^{-2} + 3C(1+0.12)^{-3} + C(1+0.12)^{-4} = 6.055C$$

$C = 36.2\$$

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:



For A :

$$(P)A = 50\$(1+0.12)^{-1} + 50\$(1+0.12)^{-2} + 100\$(1+0.12)^{-3} + 100\$(1+0.12)^{-4} = 219.232\$$$

For B :

$$(P)B = C(1+0.12)^{-1} + 3C(1+0.12)^{-2} + 3C(1+0.12)^{-3} + C(1+0.12)^{-4} = 6.055C$$

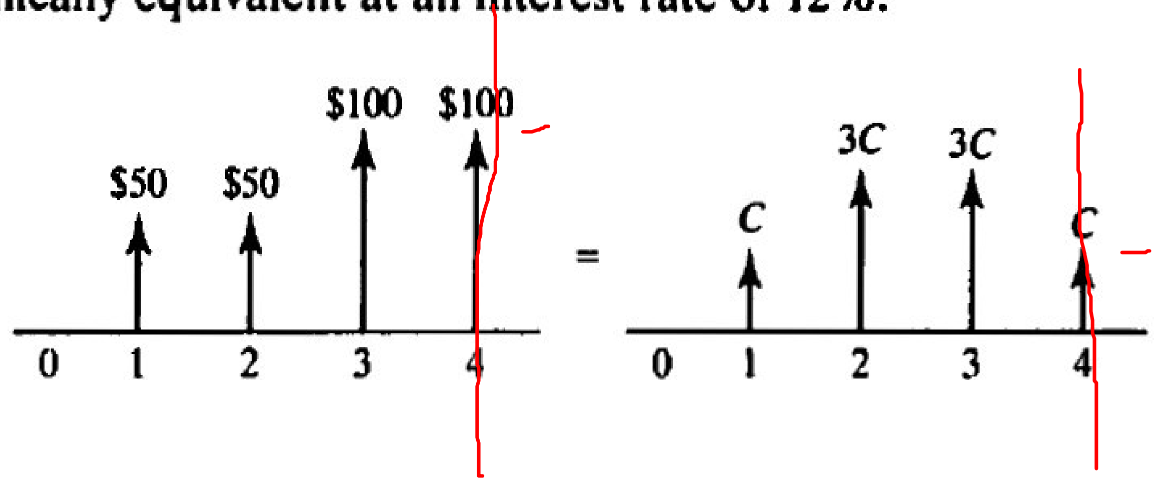
$$(P)B = (P)A$$

To Find C :

$$6.055C = 219.232\$$$

$$C = 36.2\$$$

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:

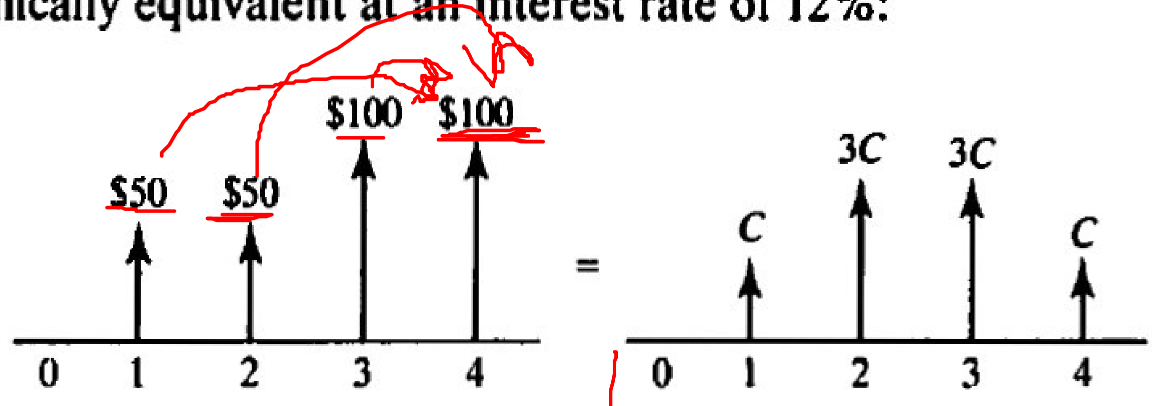


$$F = P(1+i)^n$$

Alternatively

If you choose a particular base-period, say $n = 4$, even then the answer will be same.

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:

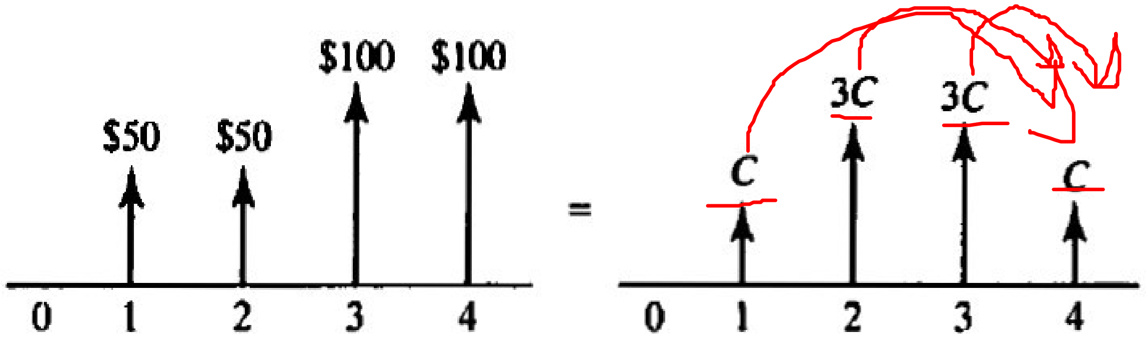


For A :

$$(V4)A = 50\$(1+0.12)^3 + 50\$(1+0.12)^2 + 100\$(1+0.12)^1 + 100\$ = \$344.96\$$$

For B :

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:



For A :

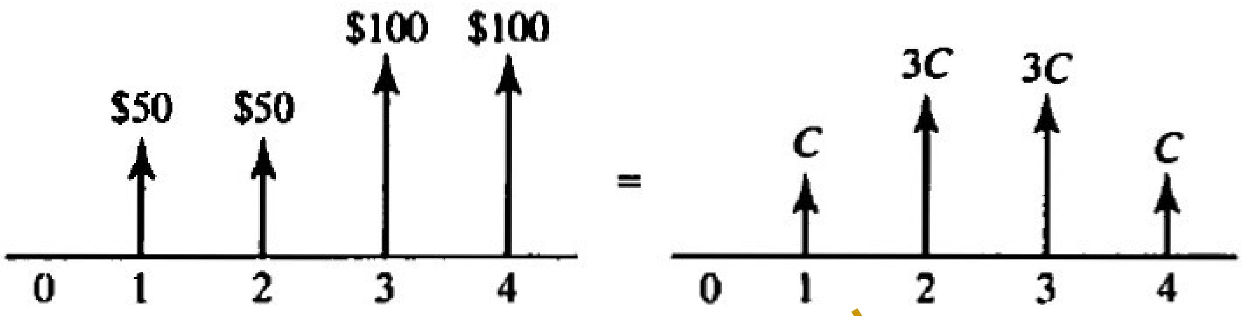
$$(V4)A = 50\$(1+0.12)^3$$
$$+50\$(1+0.12)^2$$
$$+100\$(1+0.12)^1$$
$$+100\$$$
$$= \$344.96\$$$

For B :

$$(P)B = C(1+0.12)^3$$
$$+3C(1+0.12)^2$$
$$+3C(1+0.12)^1$$
$$+C$$
$$= \underline{9.52C}$$

$$\underline{C = 36.2}$$

2.68 State the value of C that makes the following two cash flow transactions economically equivalent at an interest rate of 12%:



$V = 50(1+.12)^3+50(1+.12)^2+100(1+.12)^1+100$

$V = 70.24+62.72+112+100$

$V = \$344.96$

$V = C(1+.12)^3+3C(1+.12)^2+3C(1+.12)^1+C$

$V = 1.405C+3.76C+3.36C+C$

$V = 9.52C$

So;

$\$344.96 = 9.52C$

$C = \$36.2$

Single Cash Flow Formula

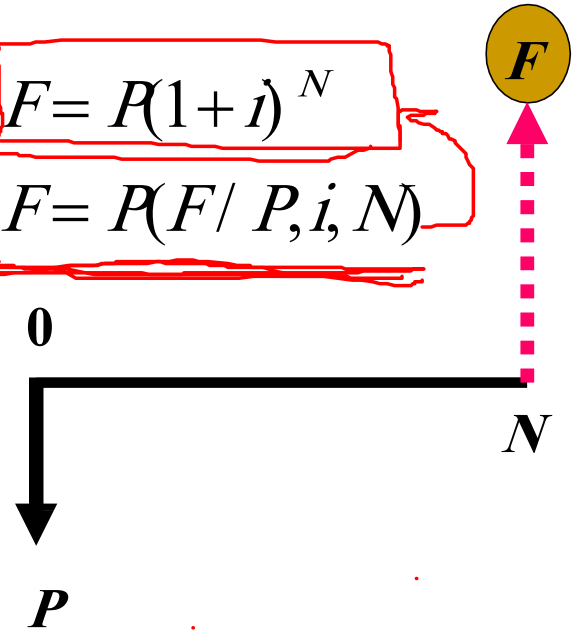
$F = P(F/P, i, N)$

- <u>Single payment compound amount factor</u> (growth factor)

Note This

$$\boxed{F = P(1+i)^N}$$

$$F = P(F/P, i, N)$$



- Given:

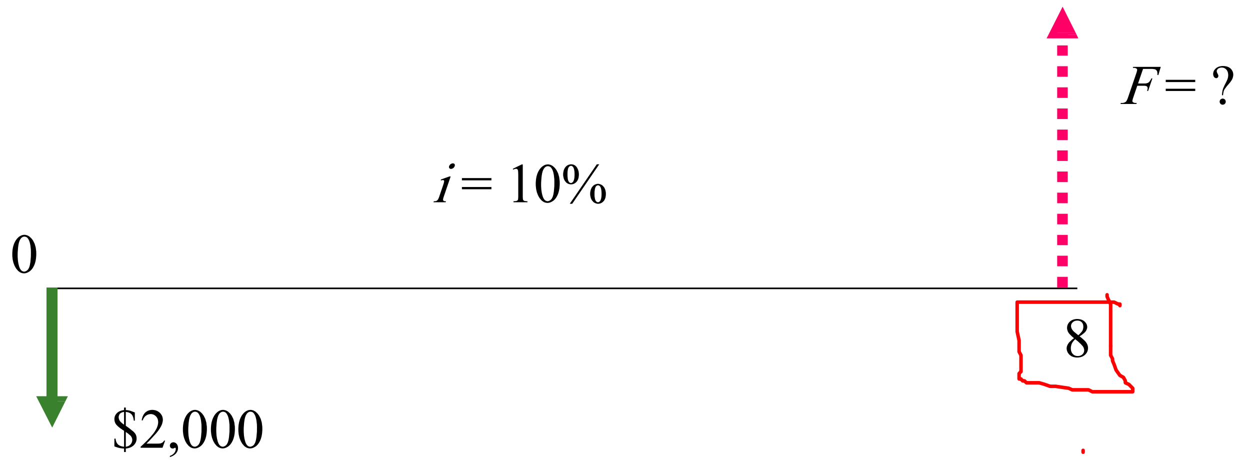
$$\begin{cases} i = 10\% \\ N = 8 \text{ years} \\ P = \$2{,}000 \end{cases}$$

- Find: F

$$\begin{aligned} F &= \$2{,}000(1+0.10)^8 \\ &= \$2{,}000(F/P, 10\%, 8) \\ &= \$4{,}287.18 \end{aligned}$$

Practice Problem

- If you had \$2,000 now and invested it at 10%, how much would it be worth in 8 years?



Solution

Given:

$$P = \$2{,}000$$
$$i = 10\%$$
$$N = 8 \text{ years}$$

Find: F

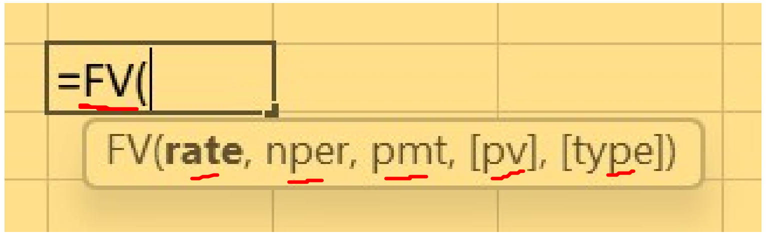
$$F = \$2{,}000(1+0.10)^8$$
$$= \$2{,}000(F/P, 10\%, 8)$$
$$= \$4{,}287.18$$

EXCEL command:

=FV(10%,8,0,2000,0)

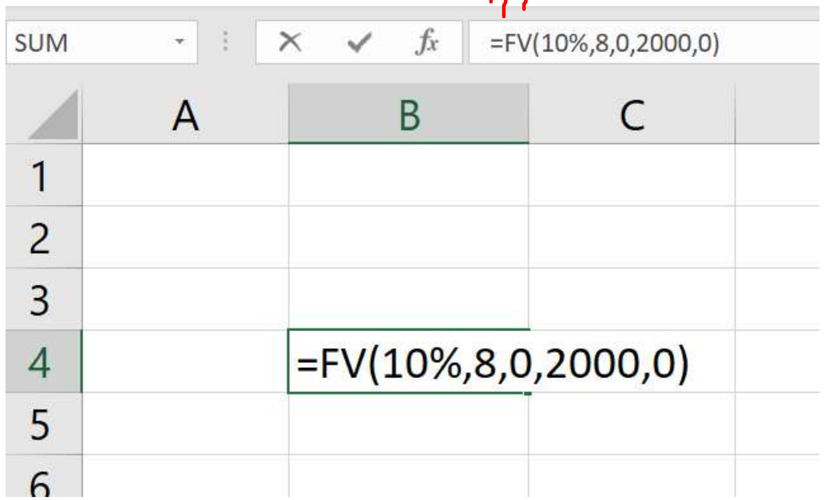
=$4,287.20

F



P

- **rate** = Interest rate per period
- **nper** = Number of periods
- **pmt** = Payment made each period
- **pv** = Present value (current lump sum) *(optional)*
- **type** = Timing of payment: 0 = end of period, 1 = beginning *(optional)*



Single Cash Flow Formula

- Single payment present worth factor (discount factor)
- Given:

$$i = 12\%$$
$$N = 5 \text{ years}$$
$$F = \$1{,}000$$

- Find P:

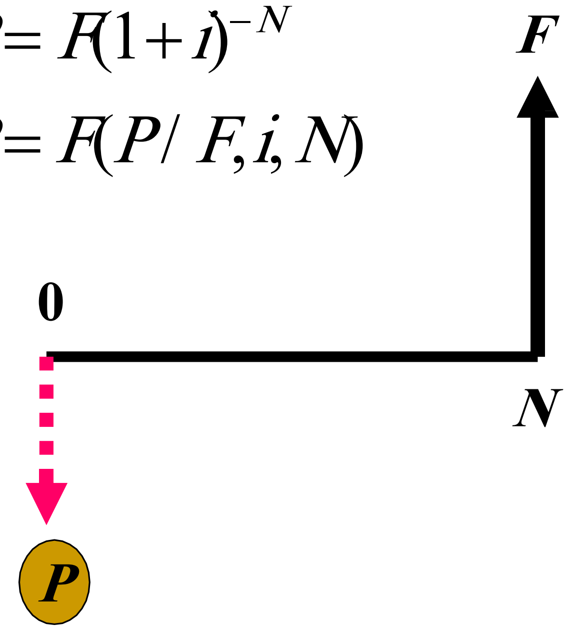
$$P = \$1{,}000(1+0.12)^{-5}$$
$$= \$1{,}000(P/F,12\%,5)$$
$$= \$567.40$$

$$\boxed{F = P(1+i)^N}$$

$$P = F(1+i)^{-N}$$

Note This

$$P = F(P/F, i, N)$$

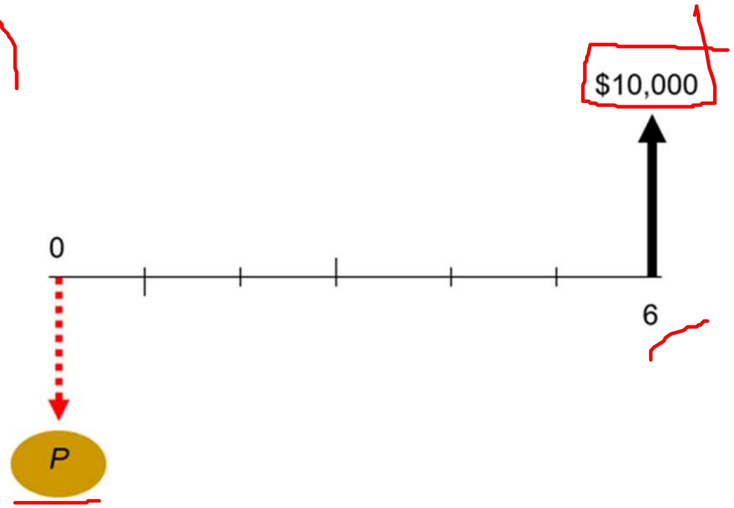


Practice Problem

- You want to set aside *a lump sum* amount today in a savings account that earns 7% annual interest to meet a future expense in the amount of $10,000 to be incurred in 6 years. How much do you need to deposit today?

$$P = F(1+i)^{-n}$$

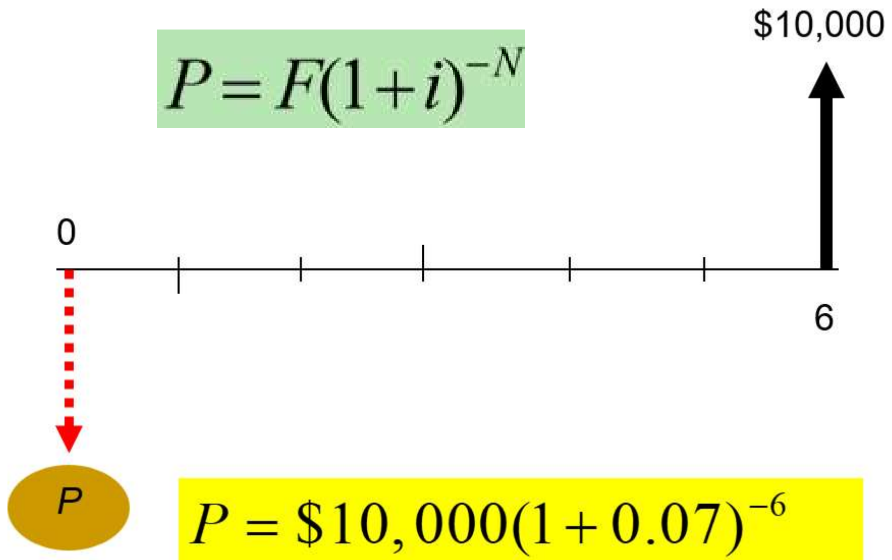
$$P = 6663\ \$$$



Practice Problem

- You want to set aside *a lump sum* amount today in a savings account that earns 7% annual interest to meet a future expense in the amount of \$10,000 to be incurred in 6 years. How much do you need to deposit today?

$$P = F(1+i)^{-N}$$



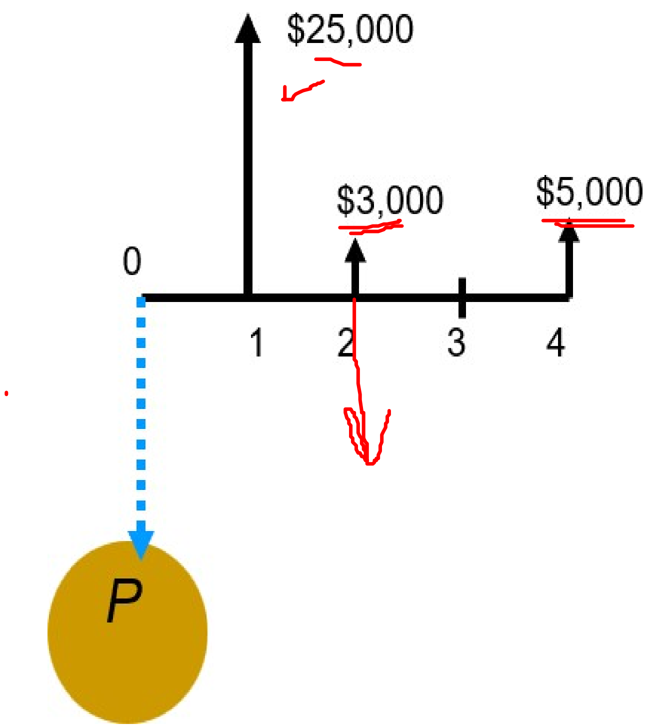
$$\begin{aligned} P &= \$10,000(1+0.07)^{-6} \\ &= \$10,000(P/F, 7\%, 6) \\ &= \$6,663 \end{aligned}$$

Multiple Payments .....(Example 2.8) Uneven Payment Series

How much do you need to deposit today (P) to withdraw \$25,000 at n=1, \$3,000 at n= 2, and \$5,000 at n=4, if your account earns 10% annual interest?

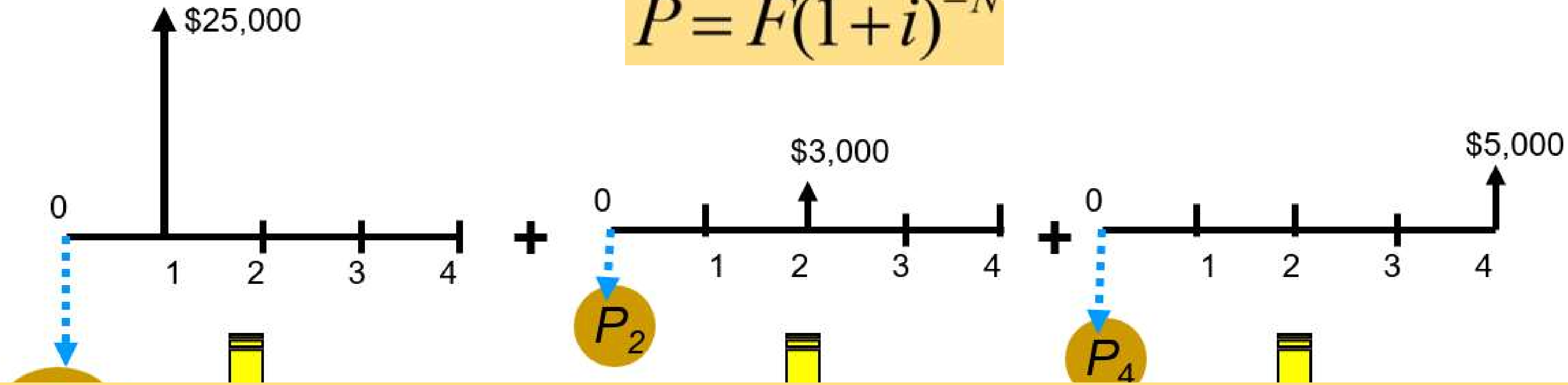
$$\boxed{P = F(1+i)^{-n}}$$

$$P_1 + P_2 + P_3$$



Multiple Payments .....(Example 2.8) Uneven Payment Series

$$P = F(1+i)^{-N}$$



This question can also be solved in a single step

P_1 = \$25000(1+0.1)$^{-1}$

$$P_1 = \$25,000(P/F,10\%,1)$$
$$= \$22,727$$

P_2 = \$3000(1+0.1)$^{-2}$

$$P_2 = \$3,000(P/F,10\%,2)$$
$$= \$2,479$$

P_4 = \$5000(1+0.1)$^{-4}$

$$P_4 = \$5,000(P/F,10\%,4)$$
$$= \$3,415$$

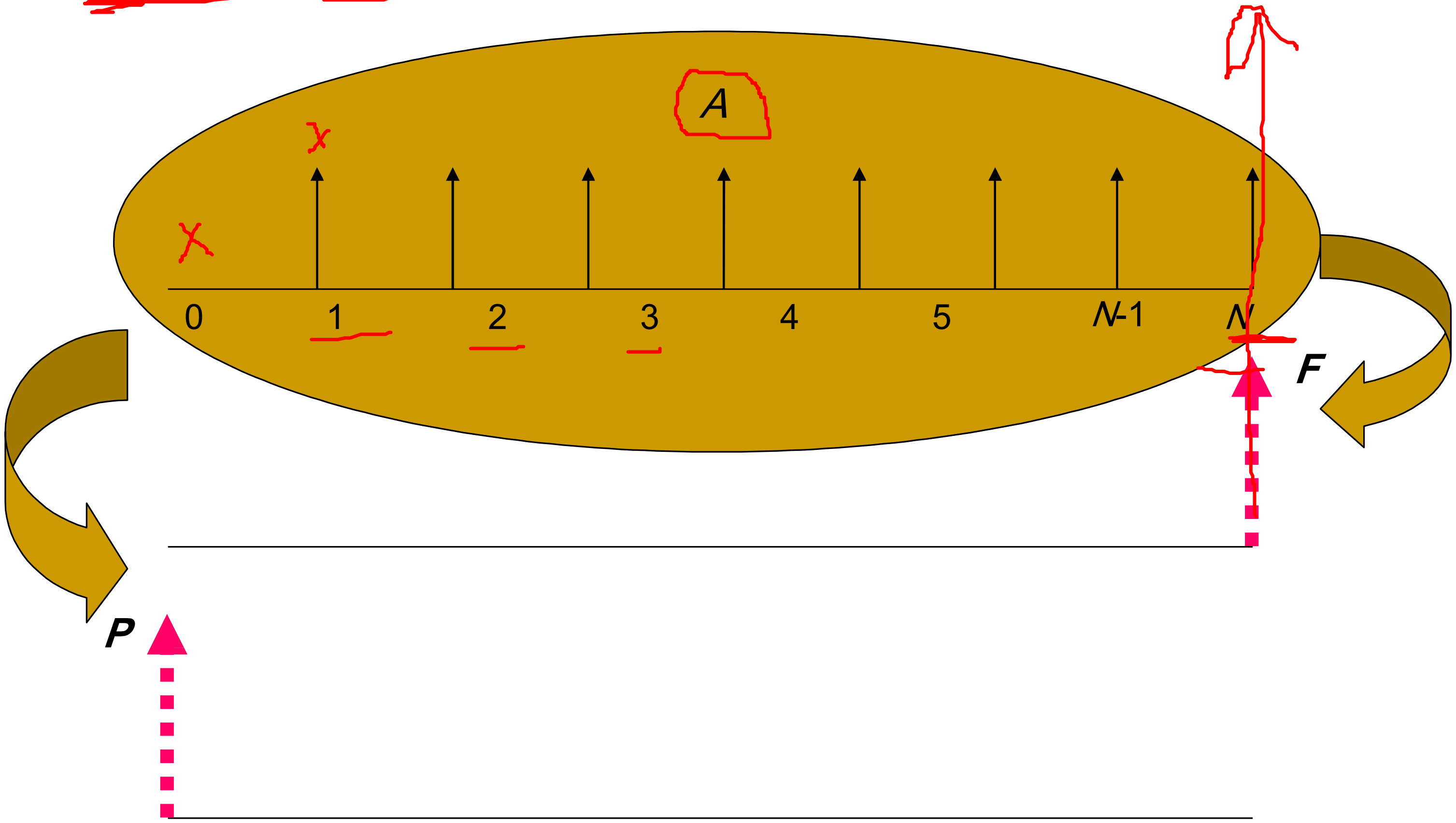
$$P = P_1 + P_2 + P_3 = \$28,622$$

CHAPTER 2

TIME VALUE OF MONEY

Uniform Series

Equal Payment Series: Find equivalent P or F



An **Equal Payment Series** refers to a series of regular, equal payments made at consistent intervals over a period of time. This concept is often used in finance to describe loan payments, savings contributions, or any other scenario where payments or receipts are made regularly.

What is the Equal Payment Series Formula?

The formula you provided:

$$\boxed{F = A\left(\frac{(1+i)^N - 1}{i}\right)}$$

is used to calculate the **Future Value (F)** of an equal series of payments (A) made at regular intervals over N periods, with an interest rate i per period.

COMPOUND-AMOUNT FACTOR $\boxed{F = P(1+i)^n}$

- A at the end of 1 year becomes = $A(1+i)^{N-1}$
  A at the end of 2 year becomes = $A(1+i)^{N-2}$

  A at the end of N-1 year becomes = $A(1+i)^1$
  A at the end of N year becomes = A

$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + \ldots\ldots\ldots\ldots\ldots + A(1+i)^1 + A$$

$$\boxed{F = A + A(1+i)^1 + \ldots\ldots\ldots\ldots\ldots + A(1+i)^{N-2} + A(1+i)^{N-1}} \quad \ldots(1)$$

$$\boxed{(1+i)F = A(1+i) + A(1+i)^2 + \ldots\ldots\ldots\ldots\ldots + A(1+i)^{N-1} + A(1+i)^N} \quad \ldots(2)$$

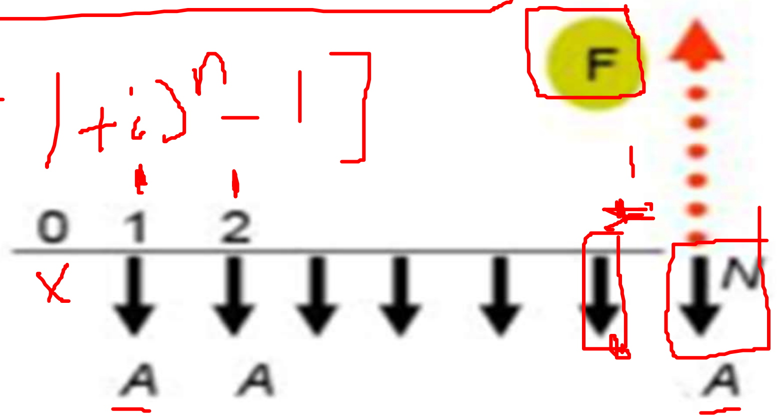
(2)-(1)

$$\boxed{(1+i)F - F = A(1+i)^N - A}$$

$F[\cancel{(1+i-1)}] = A[(1+i)^n - 1]$

$F \cdot i$

$$\boxed{F = \frac{A\{(1+i)^N - 1\}}{i}} = A(F/A, i, N)$$



$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

$$A = F\,\frac{i}{(1+i)^n - 1}$$

$$A = P(1+i)^n\,\frac{i}{(1+i)^n - 1}$$

Uniform Payment Series

Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor
Find A	Find A	Find F	Find P
Given F	Given P	Given A	Given A
A/F	A/P	F/A	P/A

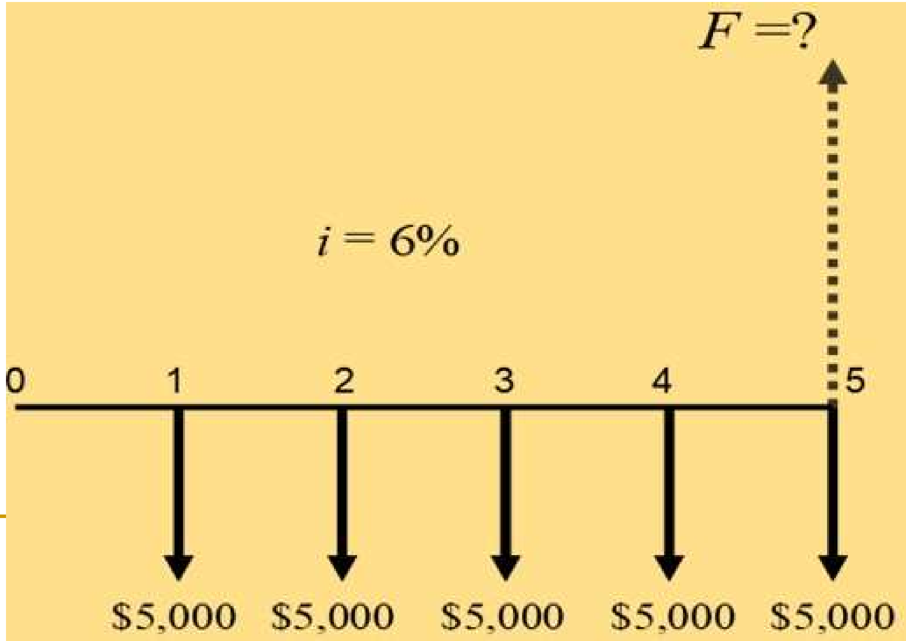
EXAMPLE 2.9 Equal-Payment Series: Find F, Given i, A, and N

Suppose you make an annual contribution of \$5,000 to your savings account at the end of each year for five years. If your savings account earns 6% interest annually, how much can be withdrawn at the end of five years? (See Figure 2.16.)

$i = 6\%$

$$F = A \frac{(1+i)^n - 1}{i}$$

$$F = 28{,}185 \ \#$$



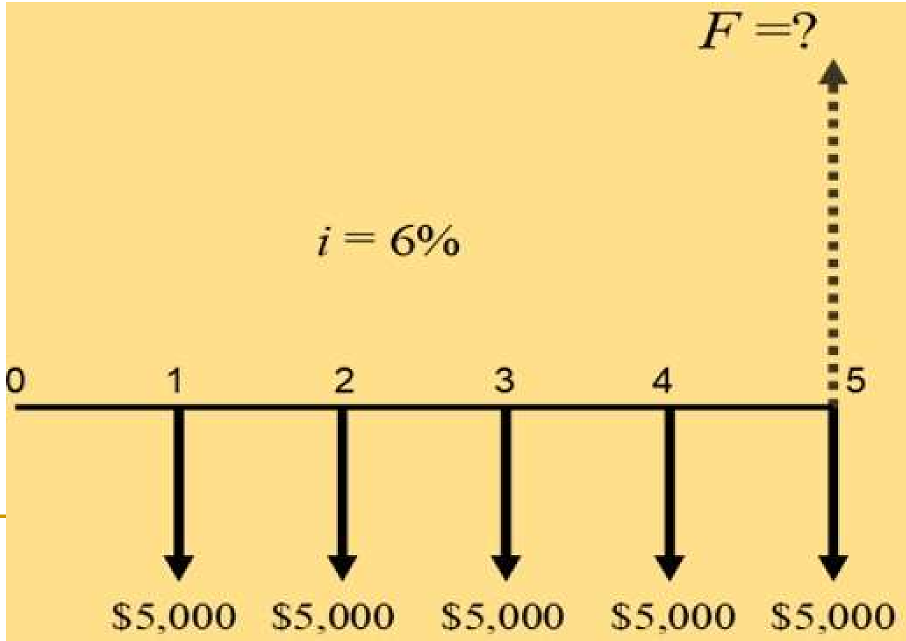
EXAMPLE 2.9 Equal-Payment Series: Find *F*, Given *i*, *A*, and *N*

Suppose you make an annual contribution of $5,000 to your savings account at the end of each year for five years. If your savings account earns 6% interest annually, how much can be withdrawn at the end of five years? (See Figure 2.16.)

$$F = A\frac{(1+i)^N - 1}{i}$$

<u>Solution</u>:

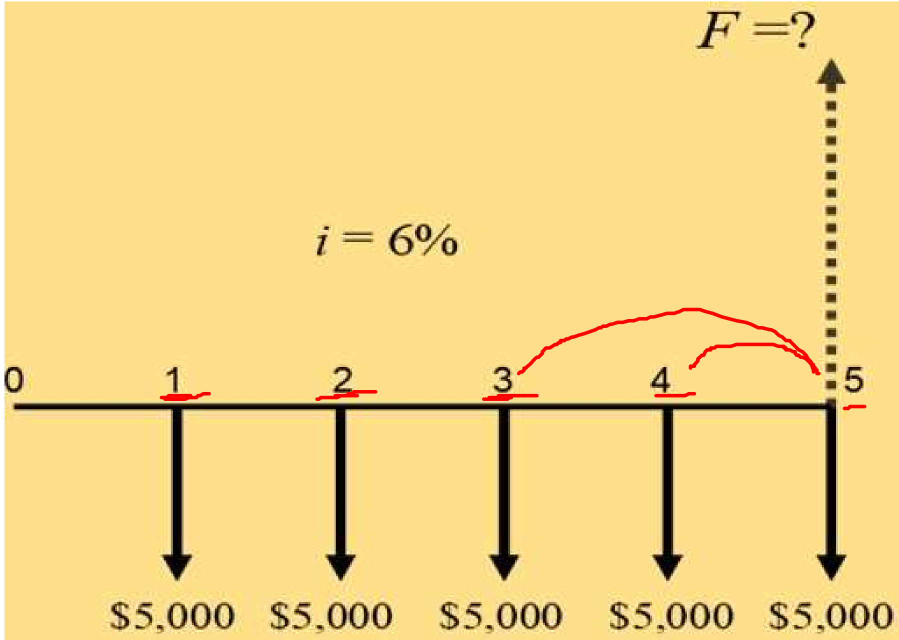
- Given: A = $5,000, N = 5 years, and i = 6%
- Find: F
- Solution: F = $5,000($F/A$, 6%, 5) = $28,185.46



Validation

$F = P(1+i)^n$

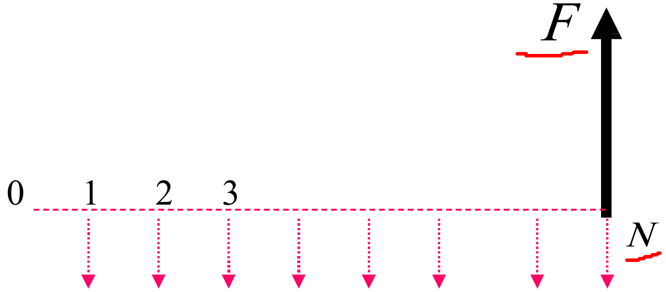
$$\$5{,}000(1+0.06)^4 = \$6{,}312.38$$
$$\$5{,}000(1+0.06)^3 = \$5{,}955.08$$
$$\$5{,}000(1+0.06)^2 = \$5{,}618.00$$
$$\$5{,}000(1+0.06)^1 = \$5{,}300.00$$
$$\$5{,}000(1+0.06)^0 = \$5{,}000.00$$
$$\$28.185.46$$



Finding an Annuity Value

Example:

- Given: $F = \$5{,}000$, $N = 5$ years, and $i = 7\%$
- Find: A



A = 869.5

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

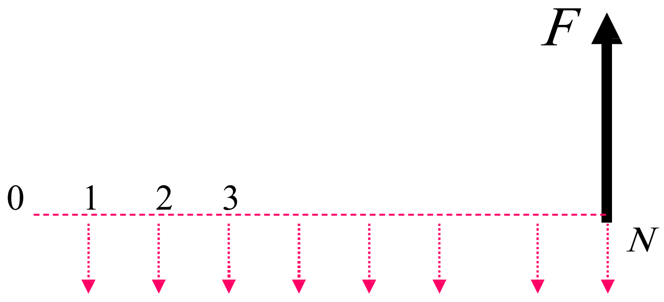
$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

Finding an Annuity Value

<u>Example</u>:

- Given: $F = \$5{,}000$, $N = 5$ years, and $i = 7\%$
- Find: A



$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

$$A = 5{,}000\left[\frac{0.07}{(1+0.07)^5 - 1}\right] = 869.5$$

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

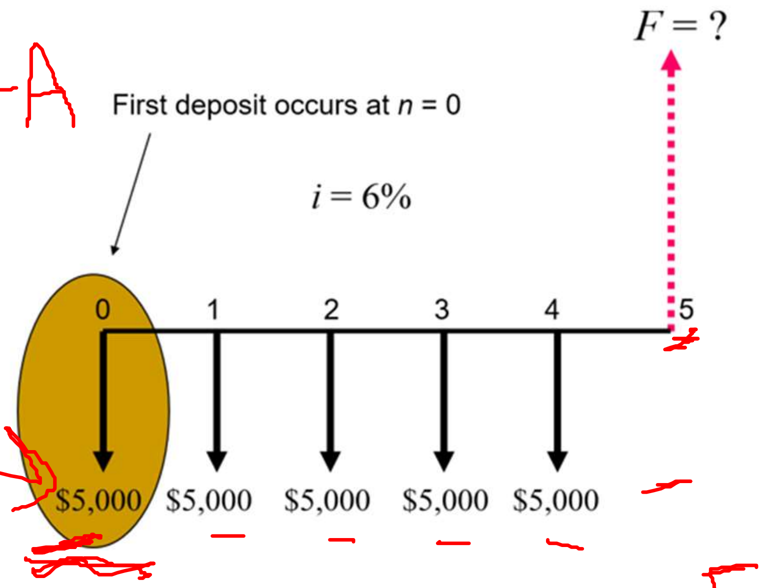
$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

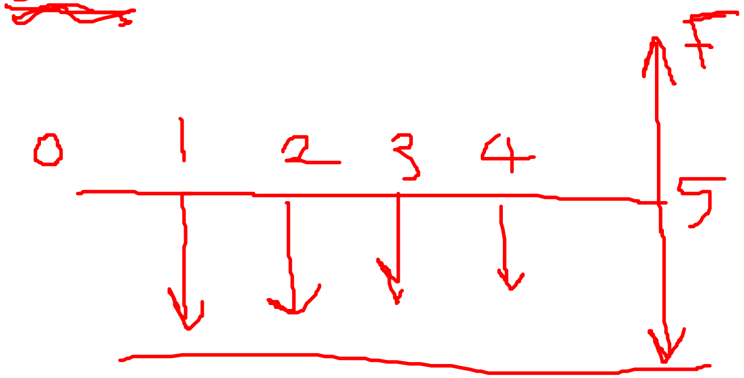
Example 2.10 Handling Time Shifts in a Uniform Series

(In the previous example, the deposits were made at the end of the each year, the first deposit being made at the end of the first year or period. suppose , all the deposits were made at the beginning of each period and commonly known as "annuity due". How would you compute the balance at the end of period 5?)

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] + P(i+1)^n - A$$



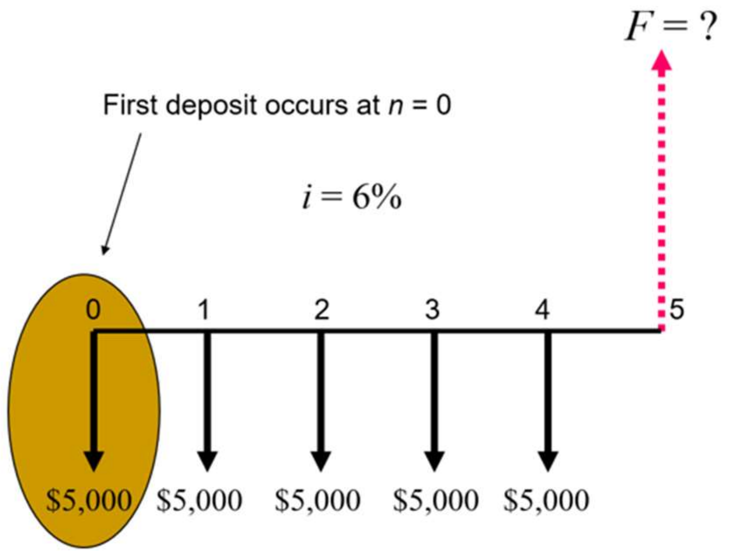
$$F = 29{,}876$$



Example 2.10 Handling Time Shifts in a Uniform Series

(In the previous example, the deposits were made at the end of the each year, the first deposit being made at the end of the first year or period. suppose , all the deposits were made at the beginning of each period and commonly known as "annuity due". How would you compute the balance at the end of period 5?)

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] \qquad F = P(1+i)^n$$



$$F = A\left[\frac{(1+i)^n - 1}{i}\right] + P(1+i)^n - A$$

$$F = 5{,}000\left[\frac{(1+0.06)^5 - 1}{0.06}\right] + 5{,}000(1+0.06)^5 - 5{,}000$$

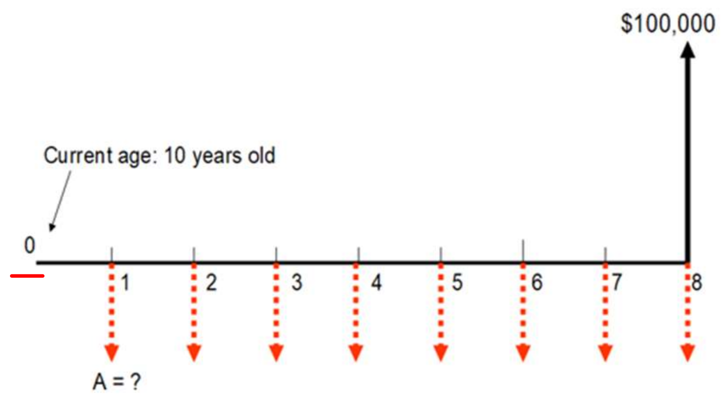
$$F = 28{,}185 + 6{,}691 - 5{,}000 = 29{,}876\$$$

EXAMPLE 2.11 College Savings Plan: Find *A*, Given *F*, *N*, and *i*

You want to set up a college savings plan for your daughter. She is currently 10 years old and will go to college at age 18. You assume that when she starts college, she will need at least \$100,000 in the bank. How much do you need to save each year in order to have the necessary funds if the current rate of interest is 7%? Assume that end-of-year deposits are made.

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

$$A = 9,746$$



EXAMPLE 2.11 College Savings Plan: Find A, Given F, N, and i

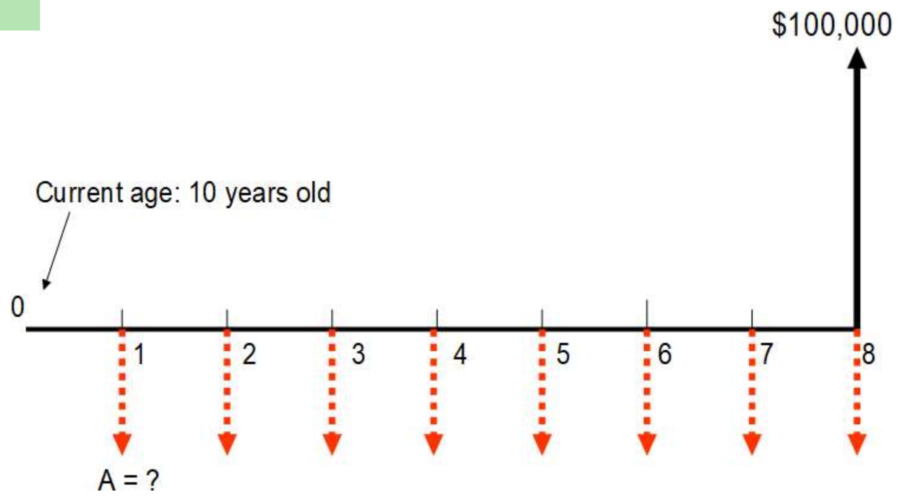
You want to set up a college savings plan for your daughter. She is currently 10 years old and will go to college at age 18. You assume that when she starts college, she will need at least \$100,000 in the bank. How much do you need to save each year in order to have the necessary funds if the current rate of interest is 7%? Assume that end-of-year deposits are made.

$$A = F \frac{i}{(1+i)^N - 1}$$

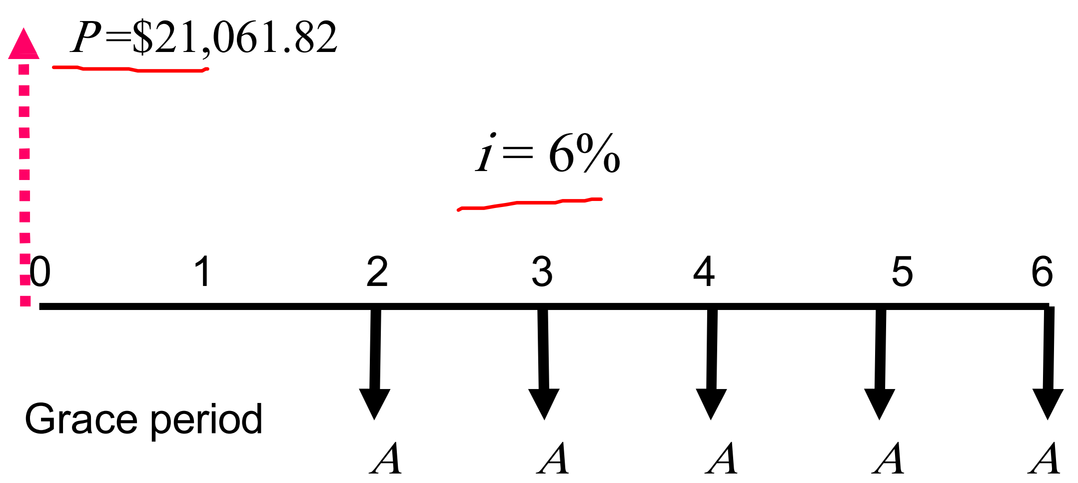
Solution:

- **Given: F = \$100,000, N = 8 years, and i = 7%**
- **Find: A**
- **Solution:**

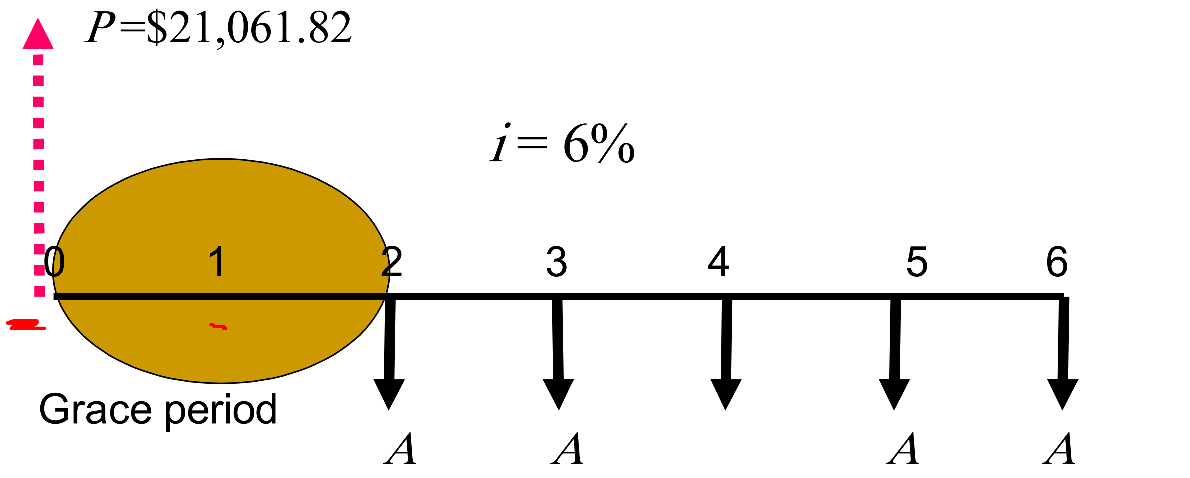
  A = \$100,000(A/F, 7%, 8) = \$9,746.78

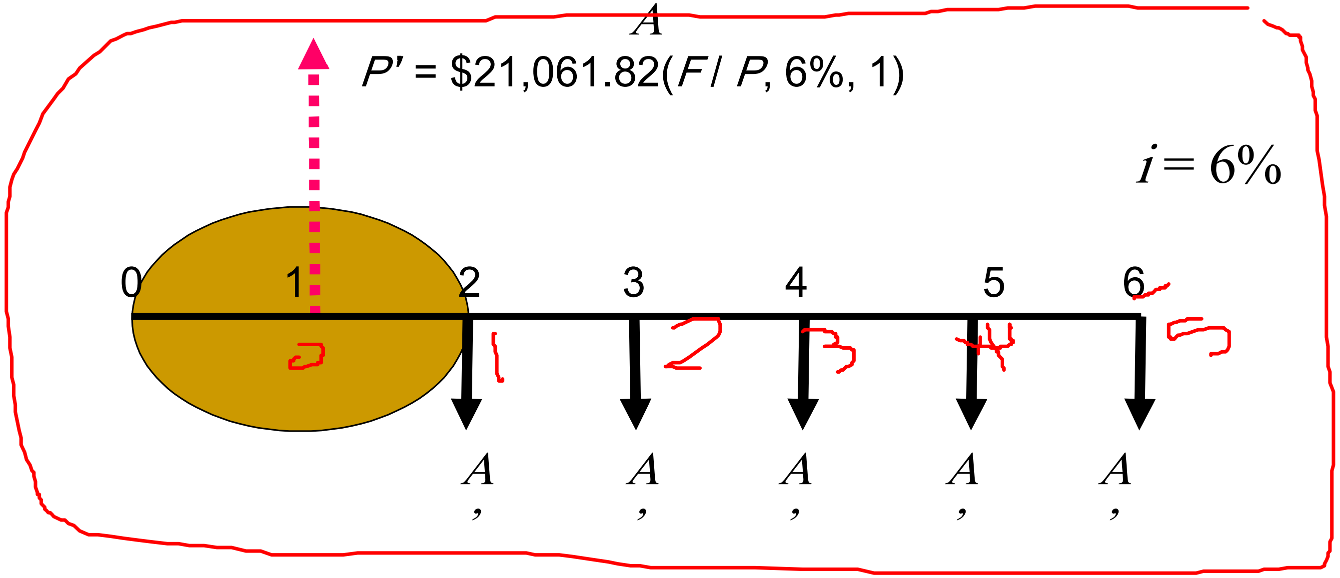


Example 2.13 Deferred (delayed) Loan Repayment Plan



Example 2.13 Deferred (delayed) Loan Repayment Plan





Example 2.13 Deferred (delayed) Loan Repayment Plan

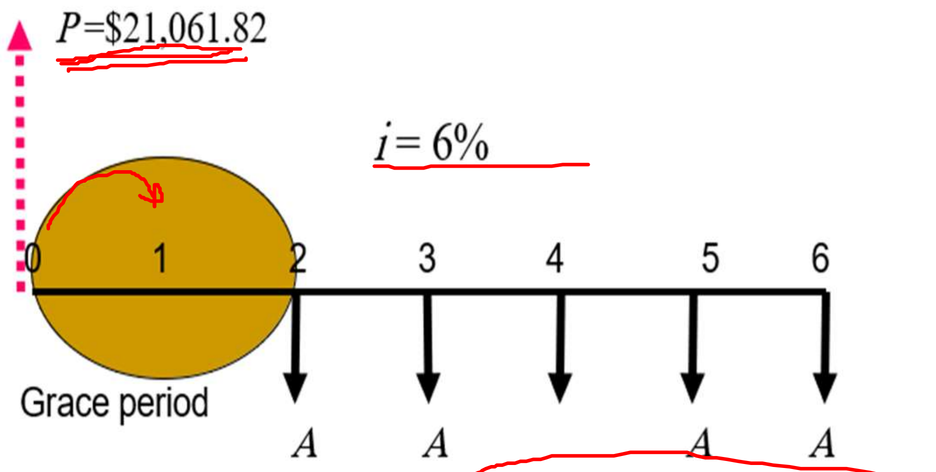
$$\boxed{F = P(1+i)^n}$$

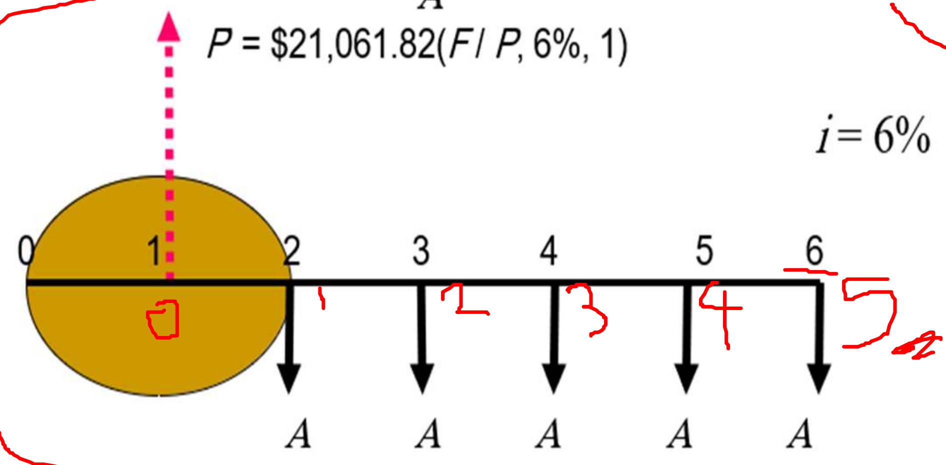
$$P' = 21{,}061.82(1+0.06)^1 = \$\,22{,}325.53$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

$$A = 22{,}325.53\left[\frac{0.06(1+0.06)^5}{(1+0.06)^5 - 1}\right]$$

$$A = \$\,5{,}300$$

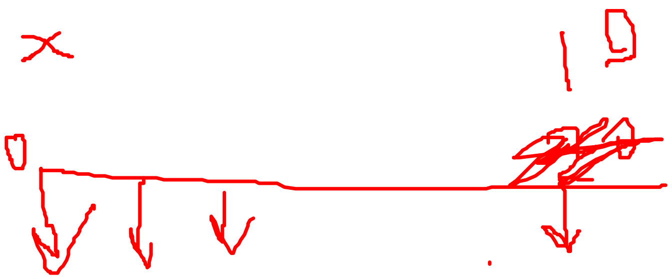




EXAMPLE 2.14 Equal-Payment Series: Find *P*, Given *A*, *i*, and *N*

A truck driver from Georgia won Powerball, a multistate lottery game similar to the one introduced in the chapter opening story. The winner could choose between a single lump sum of $116.5 million or a total of $195 million paid out over 20 annual installments (or $9.75 million per year and the first installment being paid out immediately). The truck driver opted for the lump sum. From a strictly economic standpoint, did he make the more lucrative choice?

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + A$$



$$P = 10^3$$

option 1

EXAMPLE 2.14 Equal-Payment Series: Find P, Given A, i, and N

A truck driver from Georgia won Powerball, a multistate lottery game similar to the one introduced in the chapter opening story. The winner could choose between a single lump sum of $116.5 million or a total of $195 million paid out over 20 annual installments (or $9.75 million per year and the first installment being paid out immediately). The truck driver opted for the lump sum. From a strictly economic standpoint, did he make the more lucrative choice?

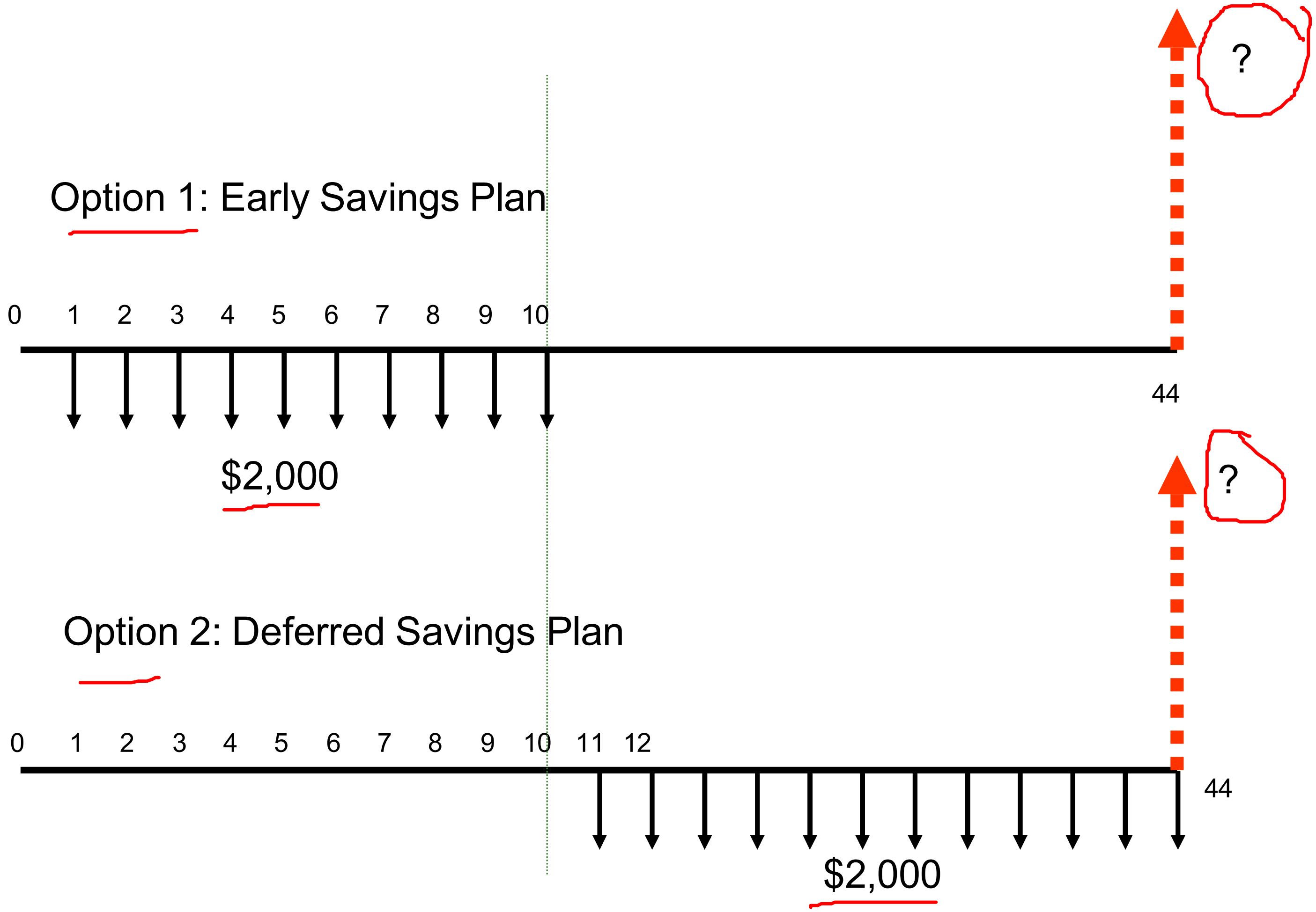
Solution:

- **Given:** A = $9.75M, N = 20 years, and i = 8%
- **Find:** P

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + A$$

$$P = \$9.75M\left[\frac{(1+0.08)^{19} - 1}{0.08(1+0.08)^{19}}\right] + \$9.75M = 103.39\text{M}$$

Example 2.15 Early Savings Plan – 8% interest



$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

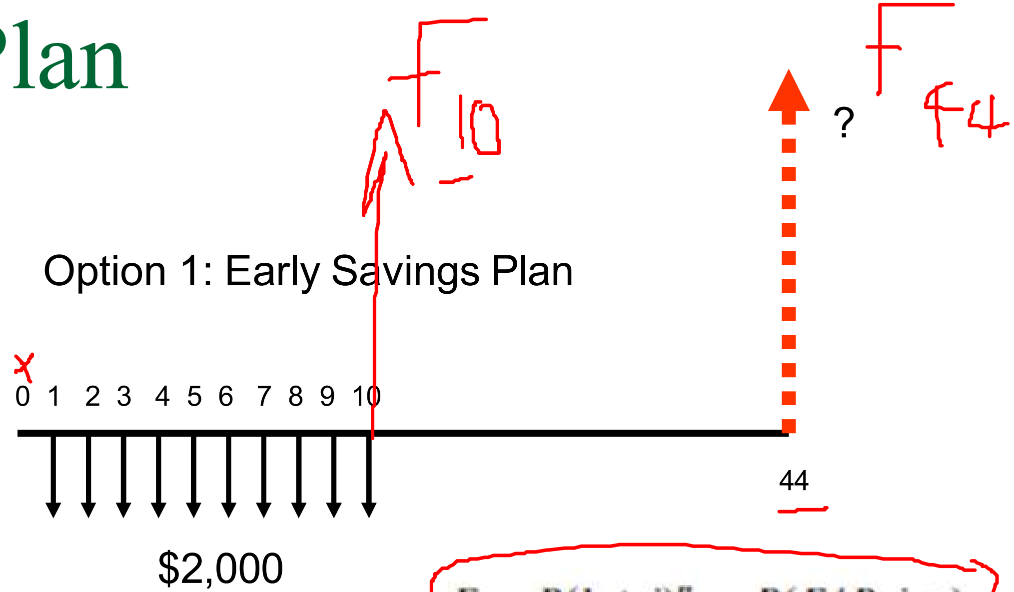
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

Option 1 – Early Savings Plan

$F_{10} = 28,973$

$F_{44} = 396,644$

Option 1: Early Savings Plan



$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

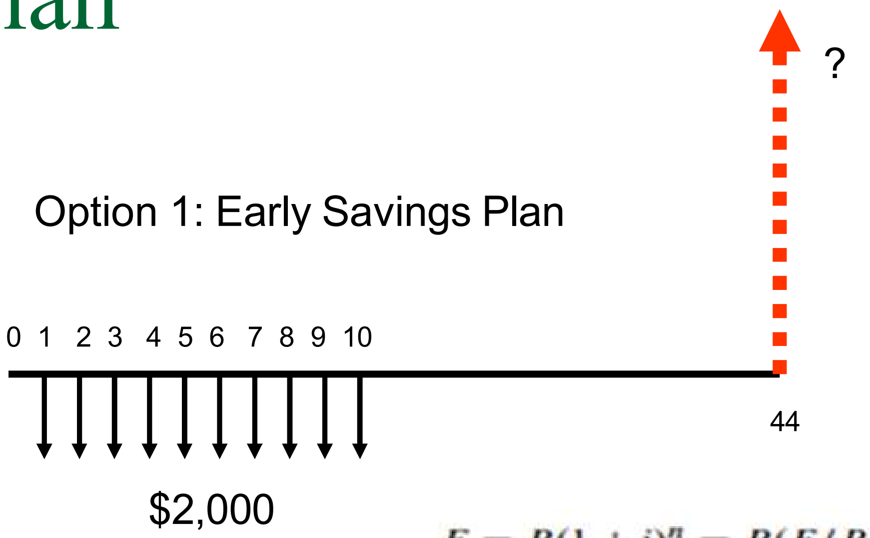
Option 1 – Early Savings Plan

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$F_{10} = 2{,}000\left[\frac{(1+0.08)^{10} - 1}{0.08}\right] = \$28,973$$

$$F = P(1+i)^n$$

$$F_{44} = \$28,973(1+0.08)^{34} = \$396,645$$



$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

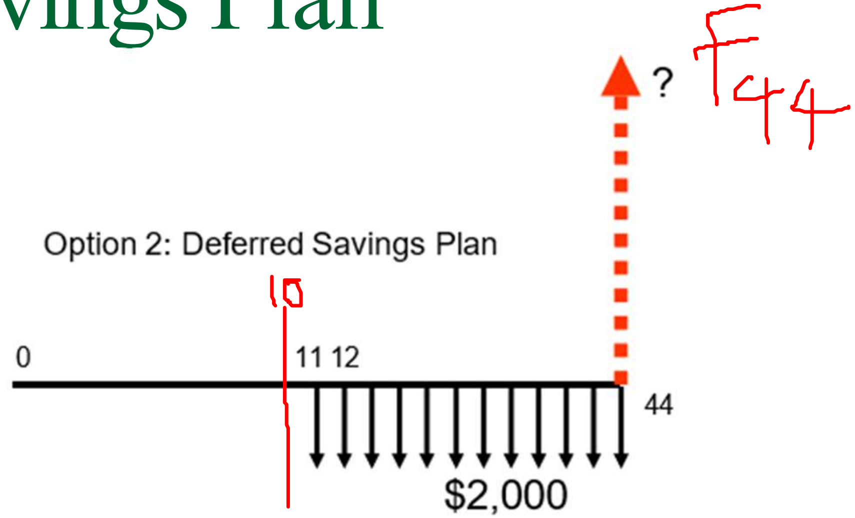
$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

Option 2: Deferred Savings Plan



$F_{44} = 317{,}253$

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

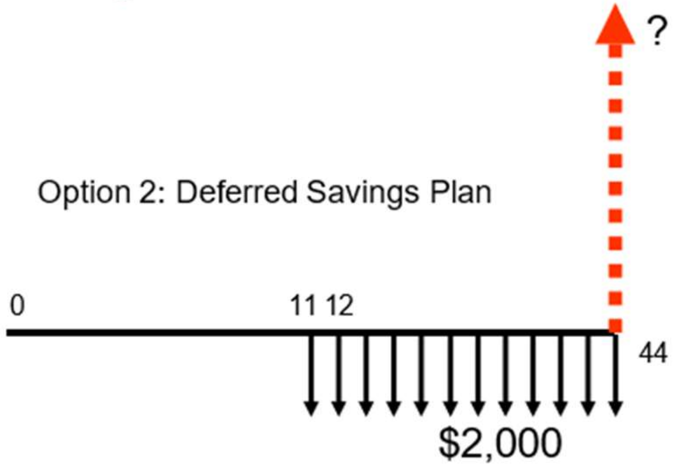
$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

Option 2: Deferred Savings Plan



$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$F_{44} = 2{,}000\left[\frac{(1+0.08)^{44-10} - 1}{0.08}\right] = \$317{,}233$$

$$F = P(1+i)^n = P(F/P, i, n)$$

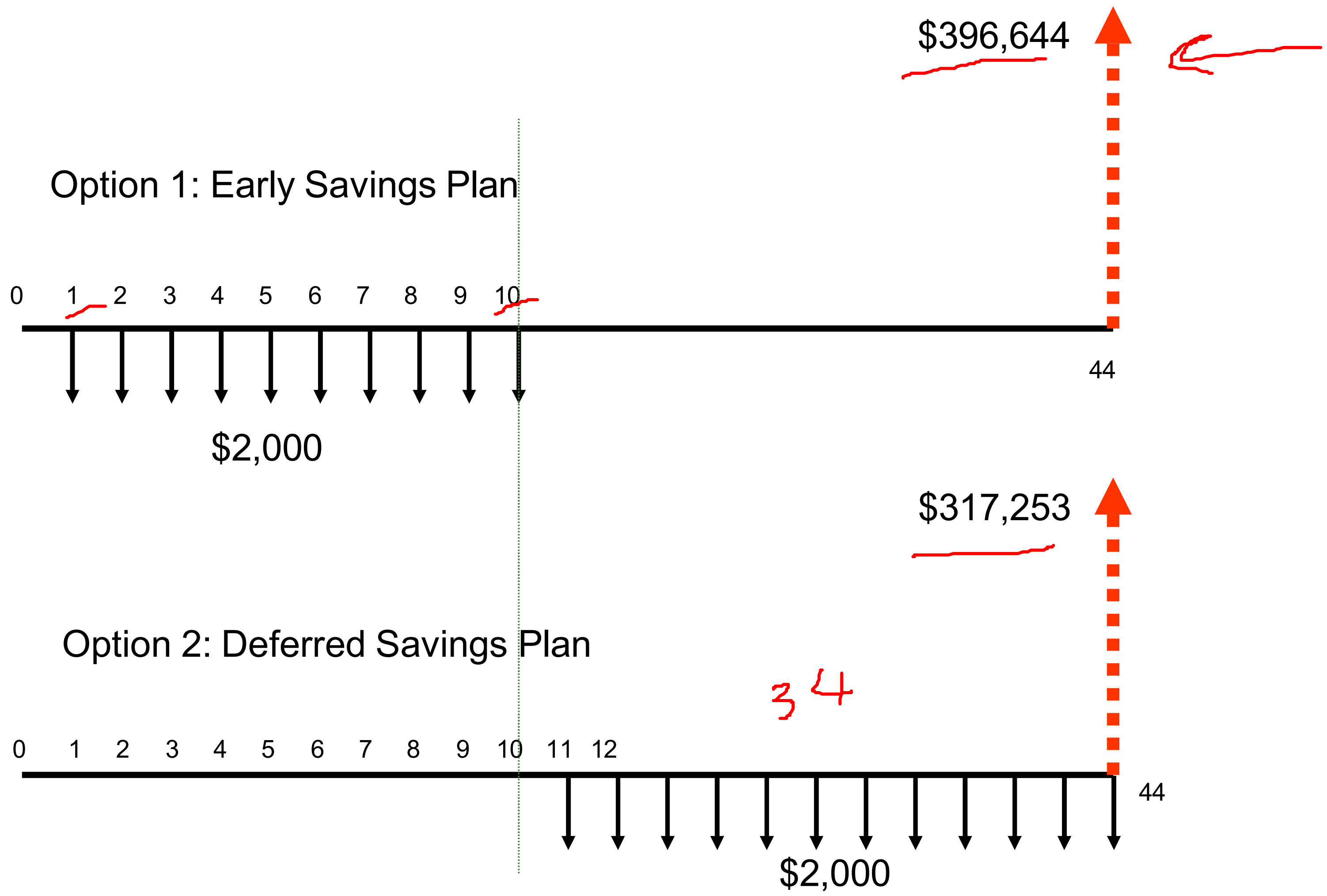
$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$


$396,644
Option 1: Early Savings Plan
0 1 2 3 4 5 6 7 8 9 10
44
$2,000
$317,253
Option 2: Deferred Savings Plan
34
0 1 2 3 4 5 6 7 8 9 10 11 12
44
$2,000

Present Value of Perpetuities

- A **perpetuity** is a stream of cash flows that continues forever. For example, a share of preferred stock that pays a fixed cash dividend each period (usually a quarter or a year) and never matures. An interesting feature of any perpetual annuity is that you can not compute the future value of its cash flows because it is infinite. However it has a well-defined present value.
- $$\boxed{P = \frac{A}{i}}$$

Example: What is the present value of $600 perpetuity at 7% discount rate?

P= A/i

PV = $600 ÷ 0.07 = $8,571.43

CHAPTER 2

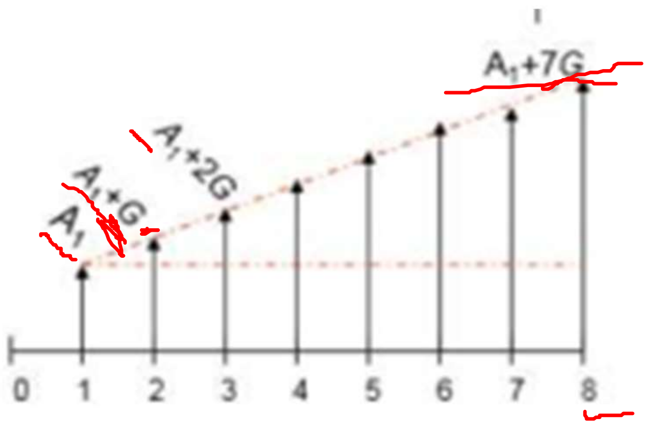
TIME VALUE OF MONEY

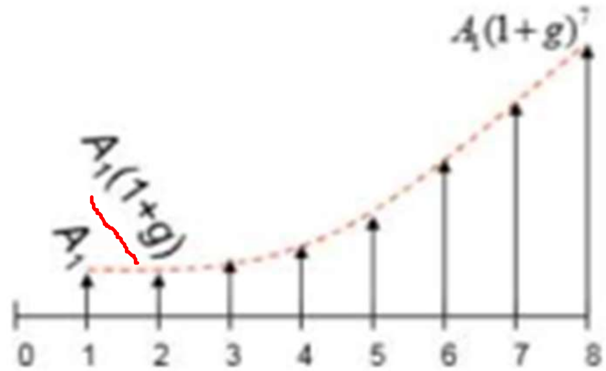
Gradient Series

Two Types of Gradient Series

- Linear Gradient
  - G – Constant $ amount

- Geometric Gradient
  - g – Constant %





Linear Gradient Series

Engineers frequently meet situations involving periodic payments that increase or decrease by a constant amount (G) from period to period.

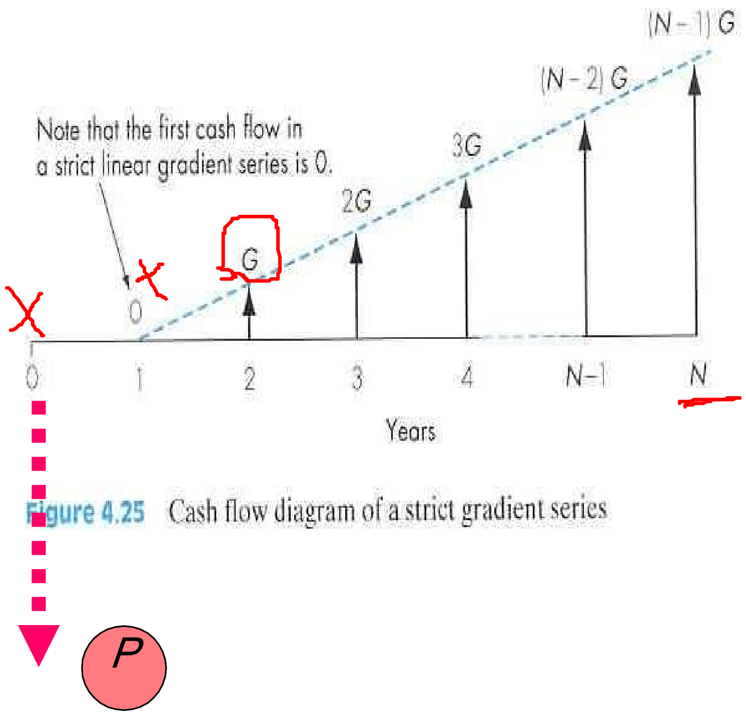
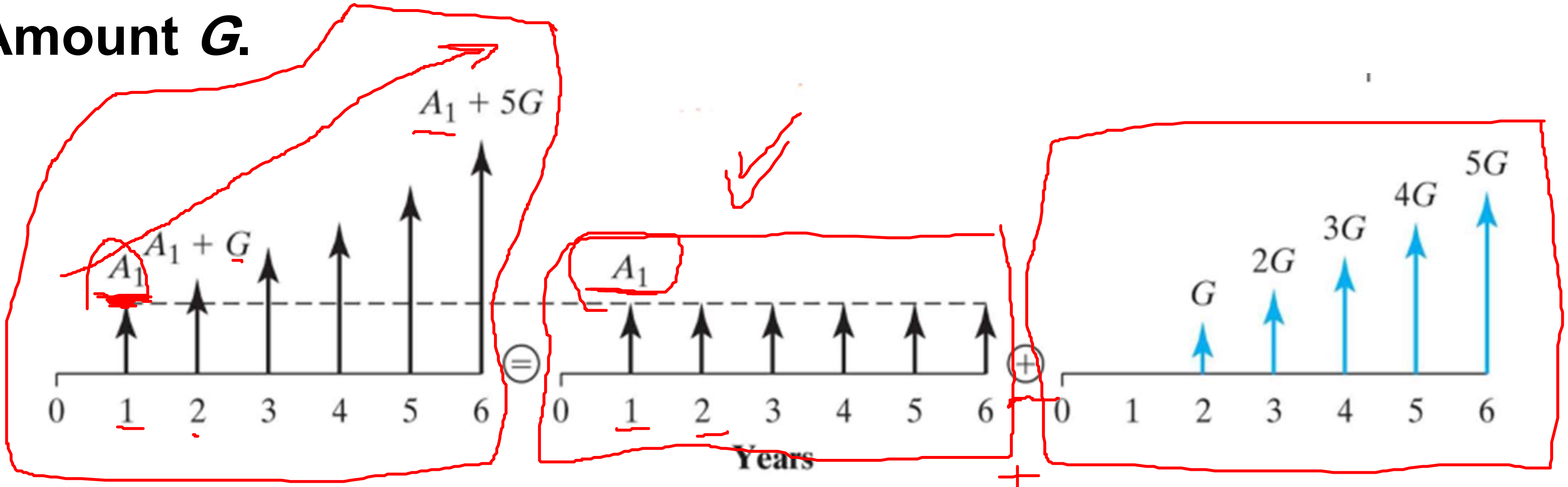


Figure 4.25 Cash flow diagram of a strict gradient series

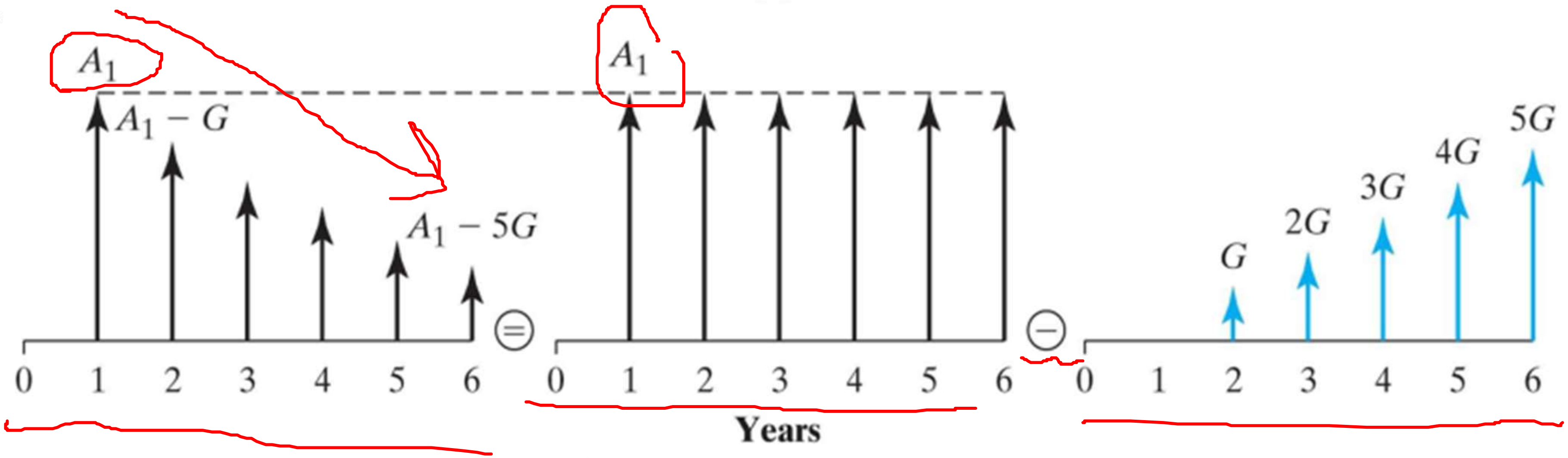
$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$$

Gradient Series as a Composite Series of a Uniform Series of N Payments of A1 and the Gradient Series of Increments of Constant Amount *G*.



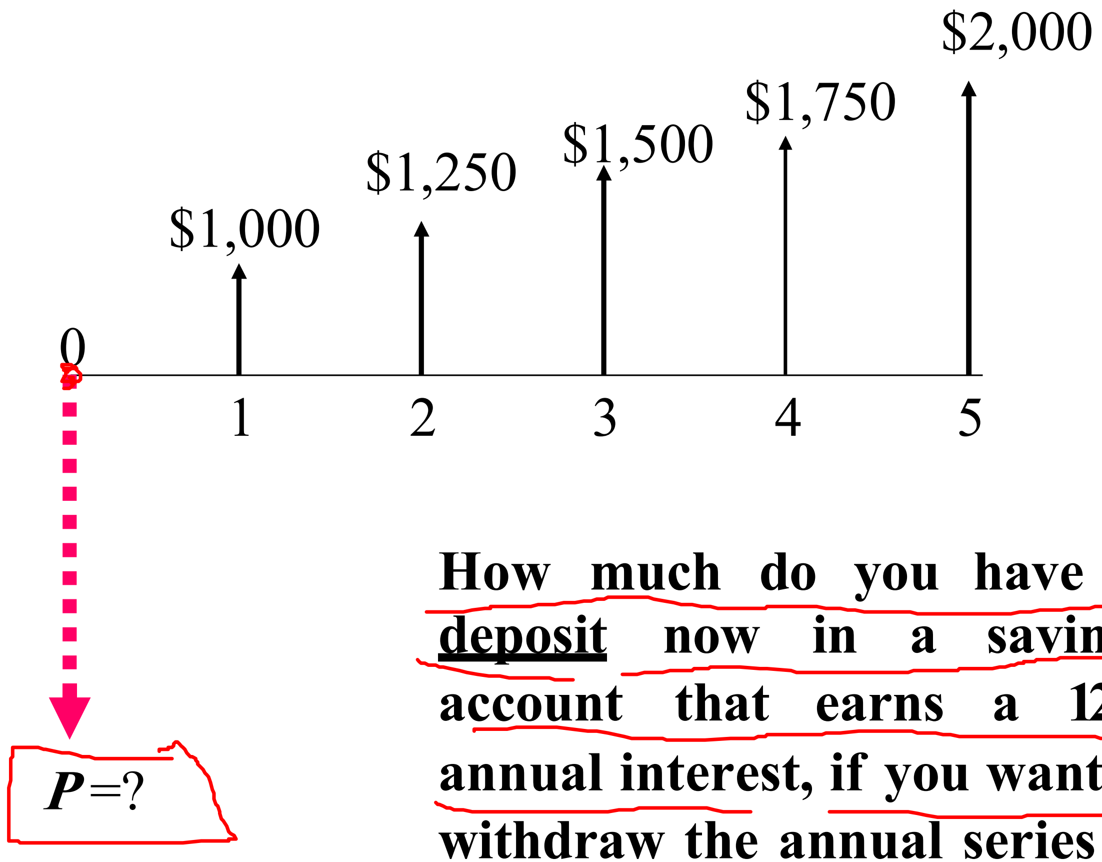
(a) Increasing gradient series

Gradient Series as a Composite Series of a Uniform Series of N Payments of A1 and the Gradient Series of Increments of Constant An



(b) Decreasing gradient series

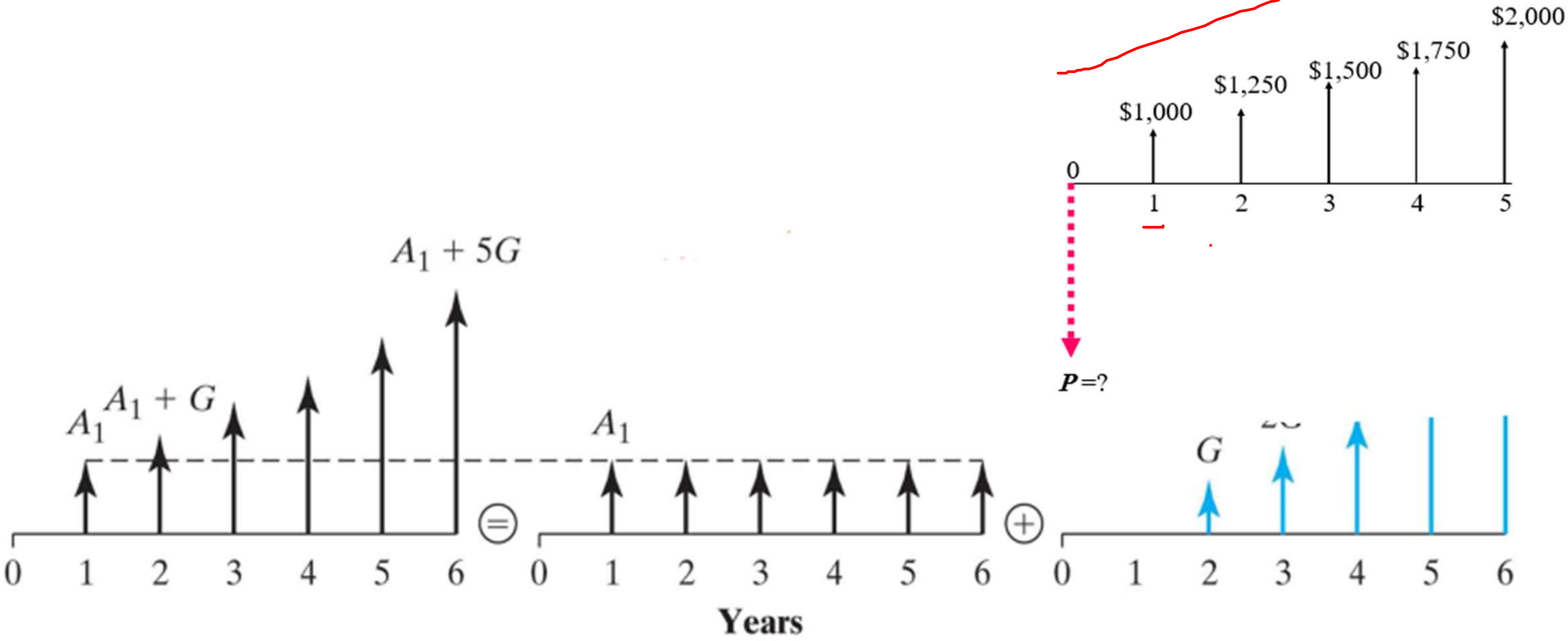
Example – Present value calculation for a gradient series



$i = 12\%$

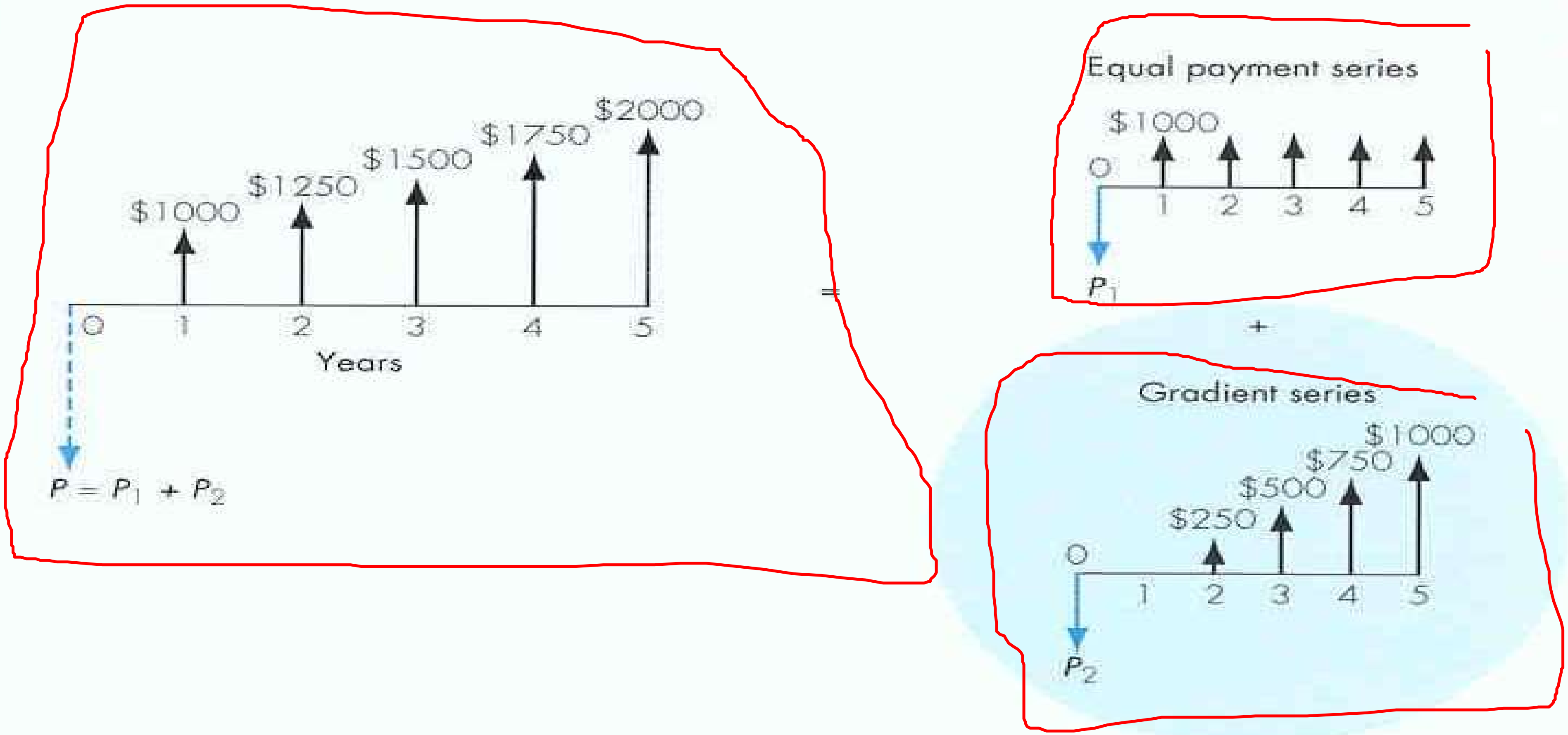
How much do you have to deposit now in a savings account that earns a 12% annual interest, if you want to withdraw the annual series as shown in the figure?

Example – Present value calculation for a gradient series



(a) Increasing gradient series

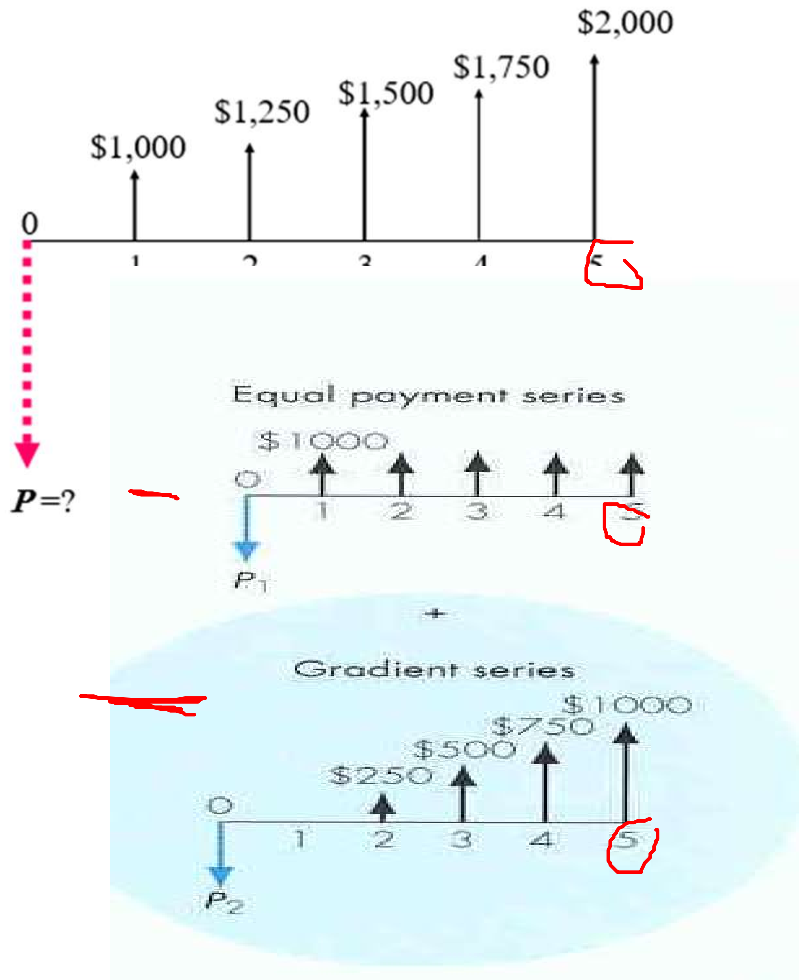
Example – Present value calculation for a gradient series



Example – Present value calculation for a gradient series

$A = 1{,}000\$ \qquad G = 250\$$

$$P = \$1{,}000(P/A, 12\%, 5) + \$250(P/G, 12\%, 5)$$



Example – Present value calculation for a gradient series

$A = 1{,}000\$$ $\qquad G = 250\$$

$$P = \$1{,}000(P/A,12\%,5) + \$250(P/G,12\%,5)$$

P = 3604 + 1599

= 5204

$$F = P(1+i)^n = P(F/P,i,n)$$

$$P = F(1+i)^{-n} = F(P/F,i,n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A,i,n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F,i,n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P,i,n)$$

1

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A,i,n)$$

2

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right] = G(P/G,i,n)$$

$$A = G\left[\frac{(1+i)^n - in - 1}{i(1+i)^n - i}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G(A/G,i,n)$$

Example – Present value calculation for a gradient series

$$A = 1{,}000\$ \qquad G = 250\$$$

$$P = \$1{,}000(\mathrm{P} / \mathrm{A}, 12\%, 5) + \$250(\mathrm{P} / \mathrm{G}, 12\%, 5)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

$$P = \$3{,}604.80 + \$1{,}599.20$$

$$P = \$5{,}204$$

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

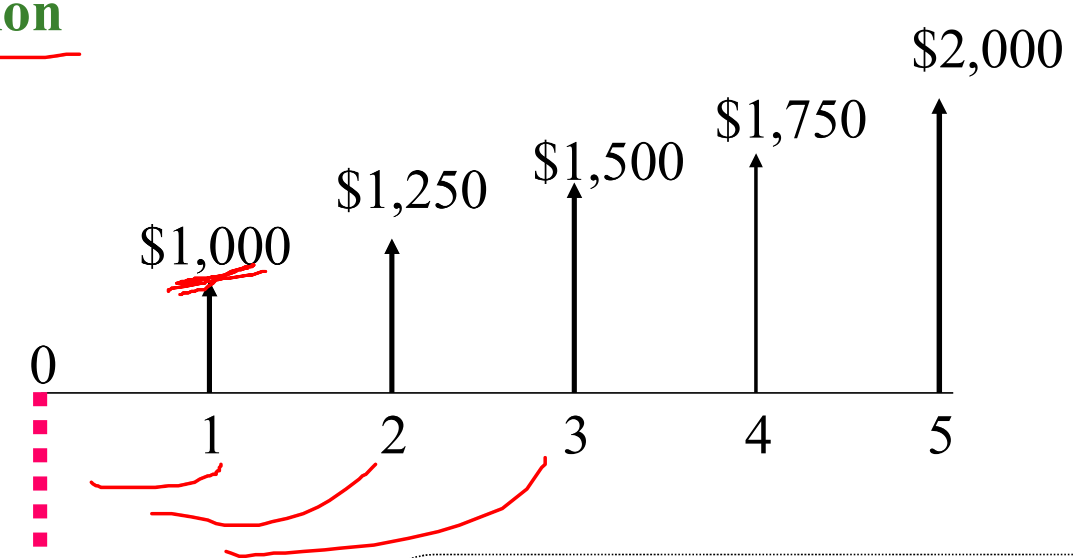
$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right] = G(P/G, i, n)$$

$$A = G\left[\frac{(1+i)^n - in - 1}{i(1+i)^n - i}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G(A/G, i, n)$$

Validation



$P = ?$

For uneven payment series

$$P = F(1+i)^{-N}$$

$$P = F(P/F, i, N)$$

$$\$1{,}000(P/F,\ 12\%,\ 1) = \$892.86$$
$$\$1{,}250(P/F,\ 12\%,\ 2) = \$996.49$$
$$\$1{,}500(P/F,\ 12\%,\ 3) = \$1{,}067.67$$
$$\$1{,}750(P/F,\ 12\%,\ 4) = \$1{,}112.16$$
$$\$2{,}000(P/F,\ 12\%,\ 5) = \$1{,}134.85$$

$5,204.03

Example 2.18 Supper Lottery

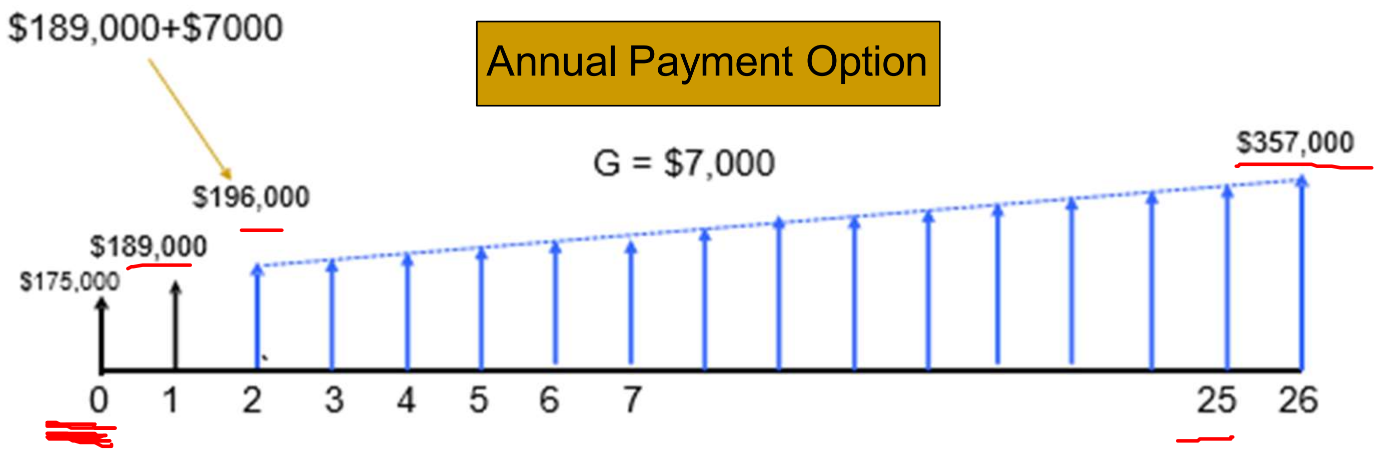
n

Annual payment option

The winner would receive the jackpot in 26 graduated annual payments. In this case, the winner would receive $175000 as the first payment(2.5% of the total jackpot amount) immediately. The second payment would be $189000. Over the course of the next 25 years, these payments would gradually increase each year by $7000 to a final payments of $357000. If the U.S. Treasury zero coupon rate is reduced to 4.5% at the time of winning, what would be the equivalent cash value of the lottery?

~~uniform~~ gr

i =

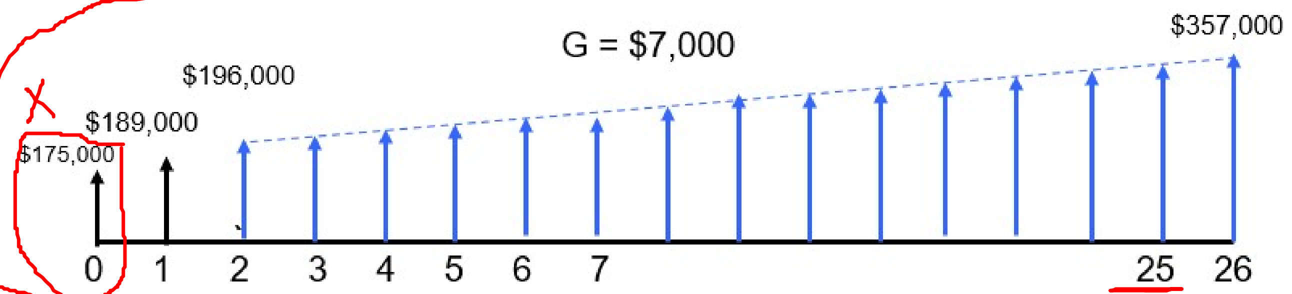


Equivalent Present Value of Annual Payment Option at 4.5%

A = 189

G = 7

Annual Payment Option



$$P = \$175{,}000 + \$189{,}000(P/A, 4.5\%, 25) + \$7000(P/G, 4.5\%, 25)$$

$$P = P_0 + A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

$$P = 175{,}000 + 2{,}802{,}531 + 1{,}012{,}658 = \$3{,}990{,}189$$

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

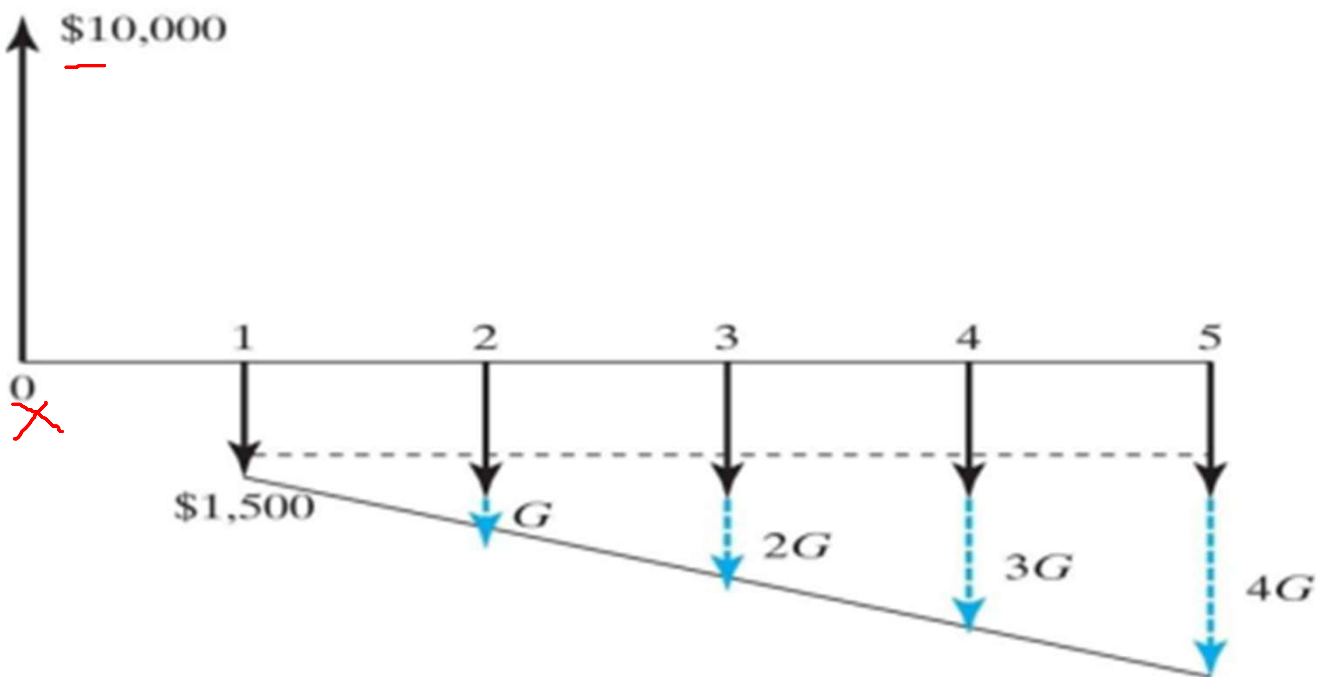
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right] = G(P/G, i, n)$$

$$A = G\left[\frac{(1+i)^n - in - 1}{i(1+i)^n - i}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G(A/G, i, n)$$

Problem :

You borrowed \$10,000 from a local bank, with the agreement that you will pay back the loan according to a graduated payment plan. If your first payment is set at \$1,500, what would the remaining payment look like at a borrowing rate of $\boxed{10\%}$ over i five years?



$P = 10,000$

$i = 10\%$

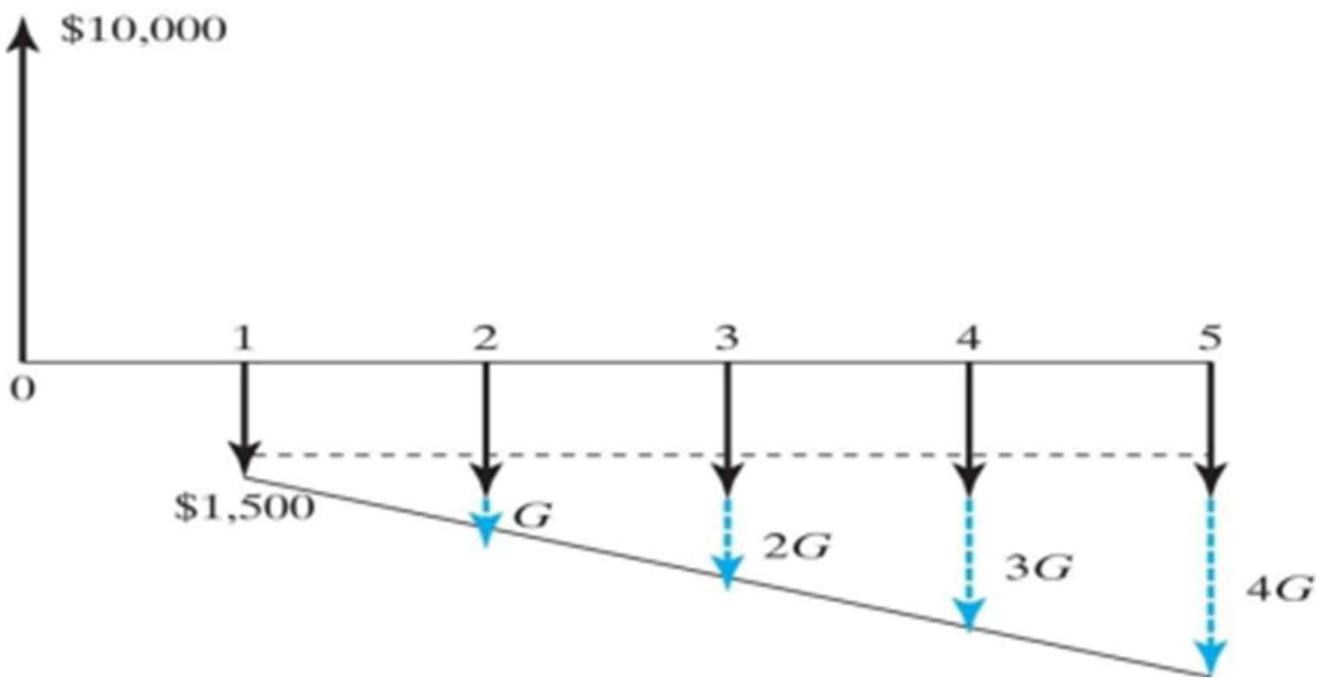
$n = 5$

$A = 1500$

$G = ?$

Problem :

You borrowed \$10,000 from a local bank, with the agreement that you will pay back the loan according to a graduated payment plan. If your first payment is set at \$1,500, what would the remaining payment look like at a borrowing rate of 10% over five years?



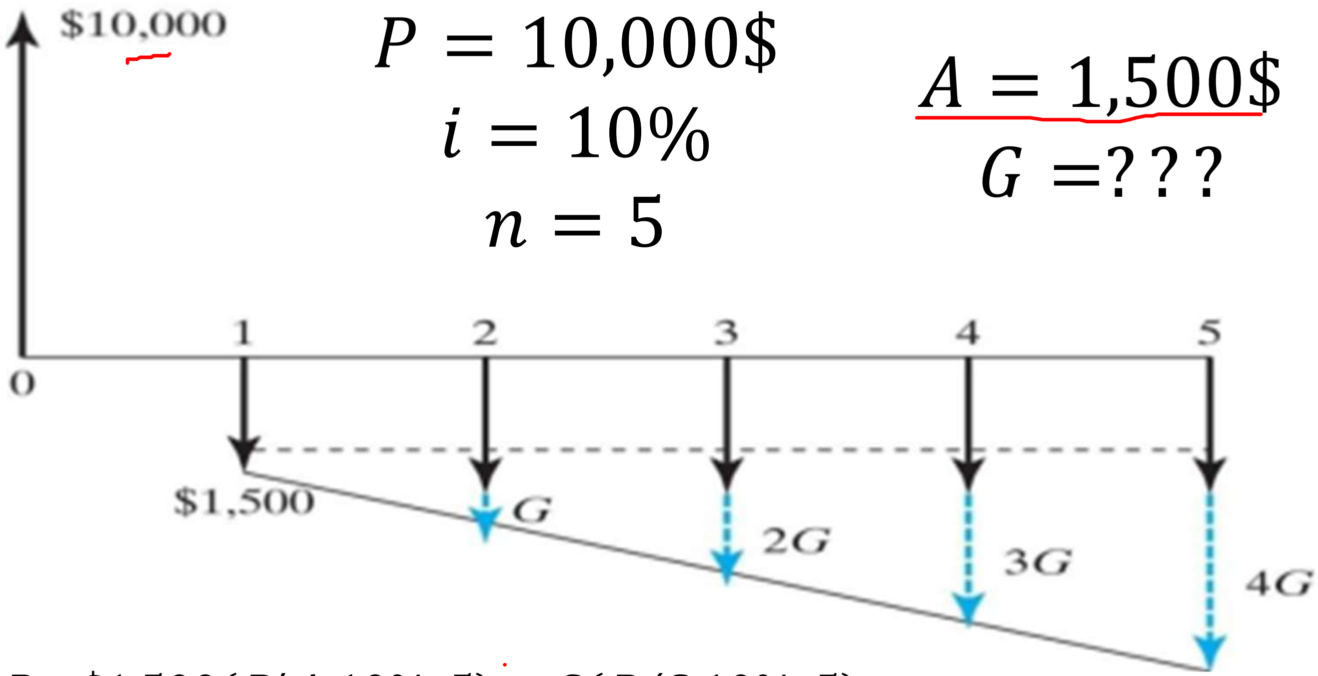
$$P = 10{,}000\$$$

$$i = 10\%$$

$$n = 5$$

$$A = 1{,}500\$$$

$$G = ???$$



$$P = 10{,}000\$$$
$$i = 10\%$$
$$n = 5$$

$$A = 1{,}500\$$$
$$G = ???$$

$$F = P(1+i)^n = P(F/P,i,n)$$

$$P = F(1+i)^{-n} = F(P/F,i,n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A,i,n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F,i,n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P,i,n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A,i,n)$$

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right] = G(P/G,i,n)$$

$$A = G\left[\frac{(1+i)^n - in - 1}{i(1+i)^n - i}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G(A/G,i,n)$$

$$P = \$1{,}500(P/A, 10\%, 5) + G(P/G, 10\%, 5)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

$$10{,}000 = 5{,}686.18 + G * 6.8618$$

$$10{,}000 - 5{,}686.18 = G * 6.8618$$

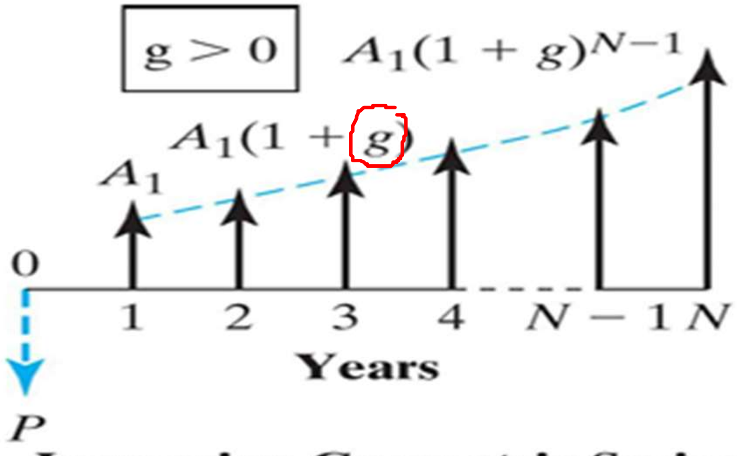
$$G = \frac{4{,}313.82}{6.8618} = \$628.67$$

Geometric Gradient Series

Many engineering economic problems, particularly those relating to construction costs, involve cash flows that increase over time, not by a constant amount, but rather by a constant percentage (geometric), called compound growth.

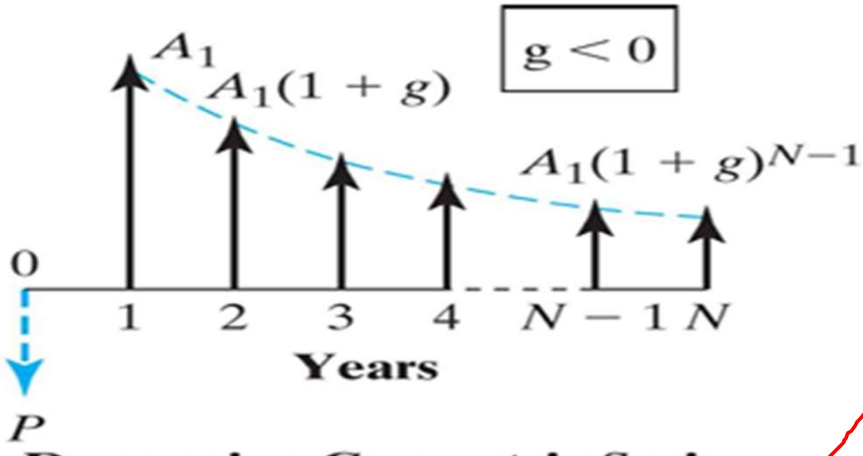
$\frac{G}{(1+g)}$

$\boxed{g+i}$



Increasing Geometric Series

or



Decreasing Geometric Series

$$\boxed{P =} \begin{cases} A_1 \dfrac{1 - (1+g)^N (1+i)^{-N}}{i-g}, & \text{if } i \neq g; \\ A_1 \dfrac{N}{(1+i)}, & \text{if } i = g \end{cases}$$

Problem

Suppose that your retirement benefits during your first year of retirement are $50000. Assume that this amount is just enough to meet your cost of living during the first year. However, your cost of living is expected to increase at an annual rate of 5% due to inflation. Suppose you do not expect to receive any cost of living adjustment in your retirement pension. If your savings account earns 7% interest a year, how much should you set aside in order to meet this future increase in the cost of living over 25 years?

Example 2.17: Find P, Given A_1, g, i, N

- Given:
  - g = 5%
  - i = 7%
  - N = 25 years
  - A_1 = \$50,000
- Find: $\boxed{\Delta P}$

ΔP

in

out

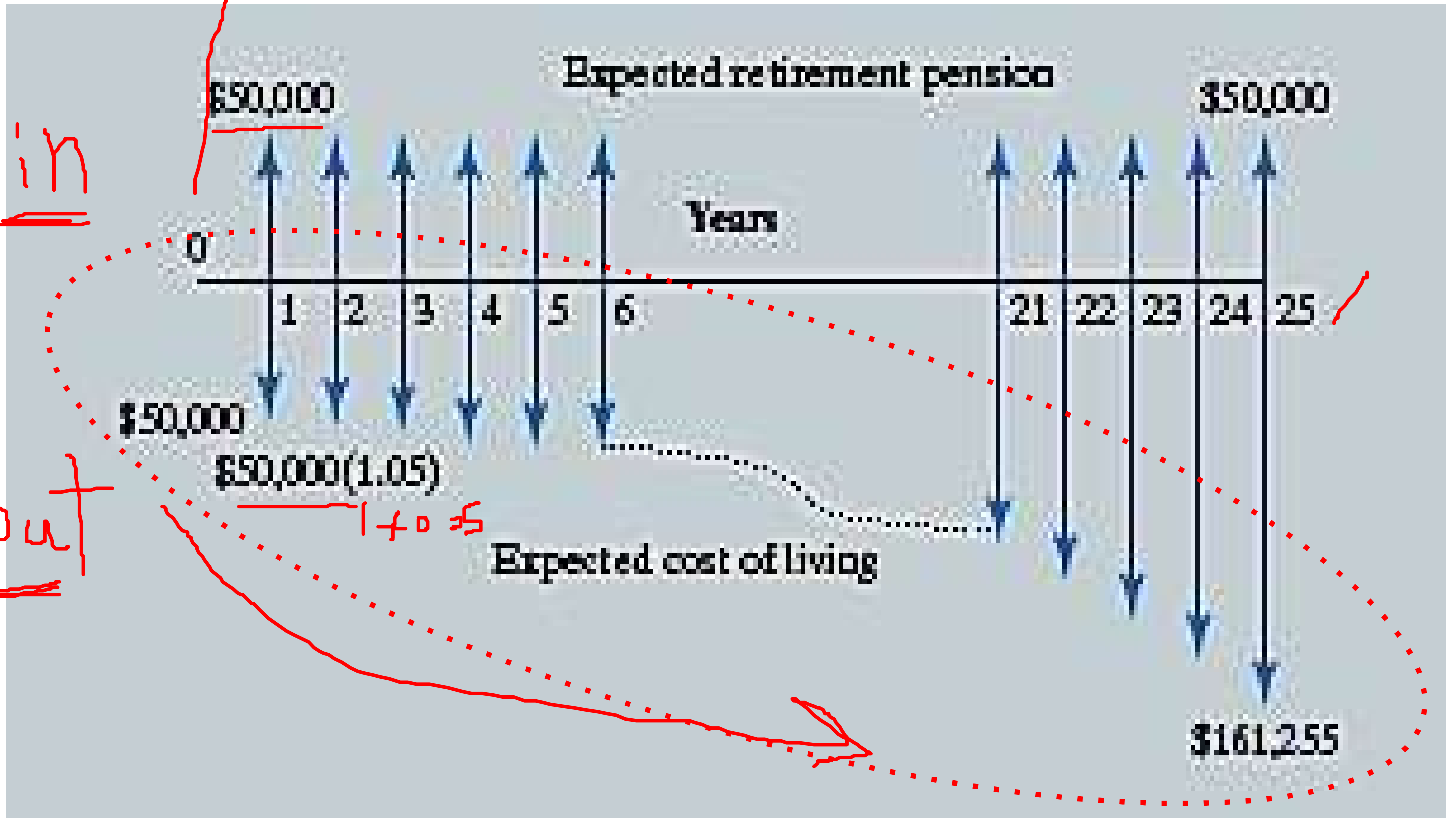


FIGURE 2.30 Cash flow diagram

EXAMPLE 9

Given: $g = 5\%$, $i = 7\%$, $N = 25$ years, $A_1 = \$50{,}000$

$$P = \begin{cases} A_1 \dfrac{1 - \left[(1+g)^N (1+i)^{-N}\right]}{i - g}, & \text{if } i \neq g; \\ A_1 \dfrac{N}{(1+i)}, & \text{if } i = g \end{cases}$$

The givens equations

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right] = G(P/G, i, n)$$

$$A = G\left[\frac{(1+i)^n - in - 1}{i(1+i)^n - i}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G(A/G, i, n)$$

EXAMPLE 9

Given: $g = 5\%$, $i = 7\%$, $N = 25$ years, $A_1 = \$50{,}000$

$$P = \begin{cases} A_1 \dfrac{1 - [(1+g)^N (1+i)^{-N}]}{i-g}, \text{ if } i \neq g; \\ A_1 \dfrac{N}{(1+i)}, \text{ if } i = g \end{cases}$$

The givens equations

$$F = P(1+i)^n = P(F/P, i, n)$$

$$P = F(1+i)^{-n} = F(P/F, i, n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A(F/A, i, n)$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] = F(A/F, i, n)$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P(A/P, i, n)$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A(P/A, i, n)$$

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right] = G(P/G, i, n)$$

$$A = G\left[\frac{(1+i)^n - in - 1}{i(1+i)^n - i}\right] = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G(A/G, i, n)$$

$$P = A\left[\frac{1 - (1+g)^n (1+i)^{-n}}{i-g}\right]$$

$$Pout = 50{,}000\left[\frac{1 - (1+0.05)^{25} (1+0.07)^{-25}}{0.07 - 0.05}\right]$$

$$Pout = 940{,}696\$$$

Required Additional Savings

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

$P_{out} = 940{,}696\$$

$P_{in} = \$50{,}000(P/A, 7\%, 25)$

$= \$582{,}679$

$\Delta P = \$940{,}696 - \$582{,}679$

$= \$358{,}017$

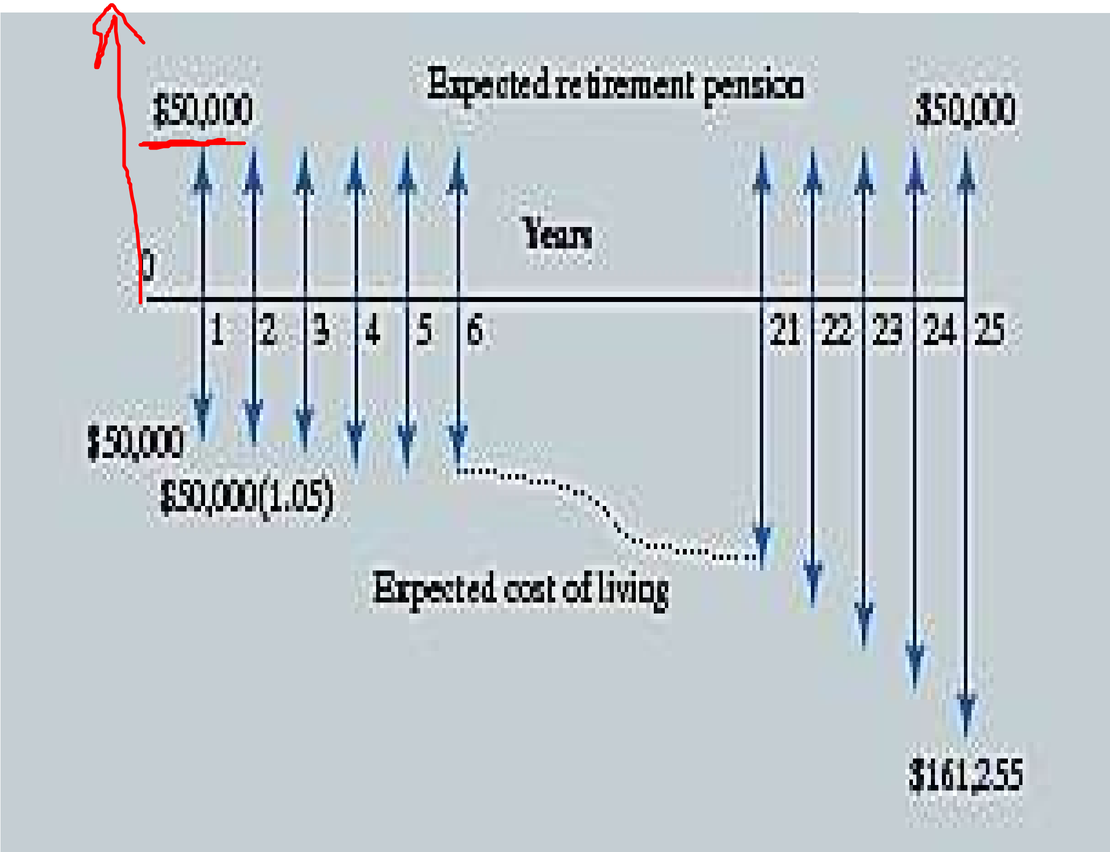


FIGURE 2.30 Cash flow diagram

CHAPTER 2

TIME VALUE OF MONEY

Thank you