System Dynamics

CHAPTER 1

# Introduction

### 1.1.1 SYSTEMS

The meaning of the term *system* has become somewhat vague because of overuse. The original meaning of the term is a *combination of elements intended to act together to accomplish an objective*. For example, a link in a bicycle chain is usually not considered to be a system. However, when it is used with other links to form a chain, it becomes part of a system. The objective for the chain is to transmit force. When the chain is combined with gears, wheels, crank, handlebars, and other elements, it becomes part of a larger system whose purpose is to transport a person.

The system designer must focus on how all the elements act together to achieve the system's intended purpose, keeping in mind other important factors such as safety, cost, and so forth. Thus, the system designer often cannot afford to spend time on the details of designing the system elements. For example, our bicycle designer might not have time to study the metallurgy involved with link design; that is the role of the chain designer. All the systems designer needs to know about the chain is its strength, its weight, and its cost, because these are the factors that influence its role in the system.

With this "systems point of view," we focus on how *connections* between the elements influence the *overall* behavior of the system. This means that sometimes we must accept a less-detailed description of the operation of the individual elements to achieve an overall understanding of the system's performance.

Figure 1.1.1 illustrates a liquid-filled tank with a volume inflow $f$ (say in cubic feet per second). The liquid height is $h$ (say in feet). We see in Example 1.4.2 that the functional relationship between $f$ and $h$ has the form $f = bh^m$, where $b$ and $m$ are constants. We would not call this a "system." However, if two tanks are connected as shown in Figure 1.1.2, this connection forms a "system." Each tank is a "subsystem"

**Figure 1.1.1** The effect of liquid height $h$ on the out flow rate $f$.

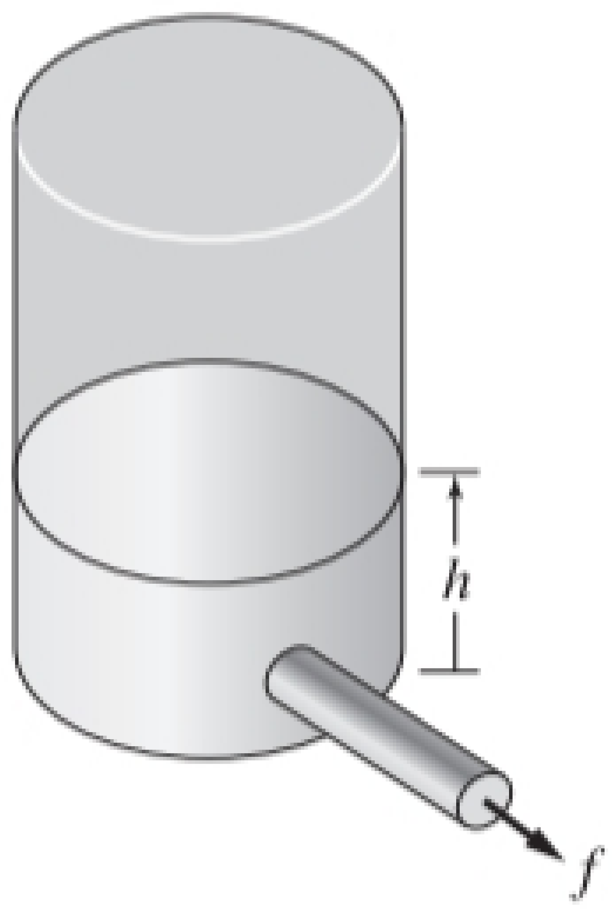
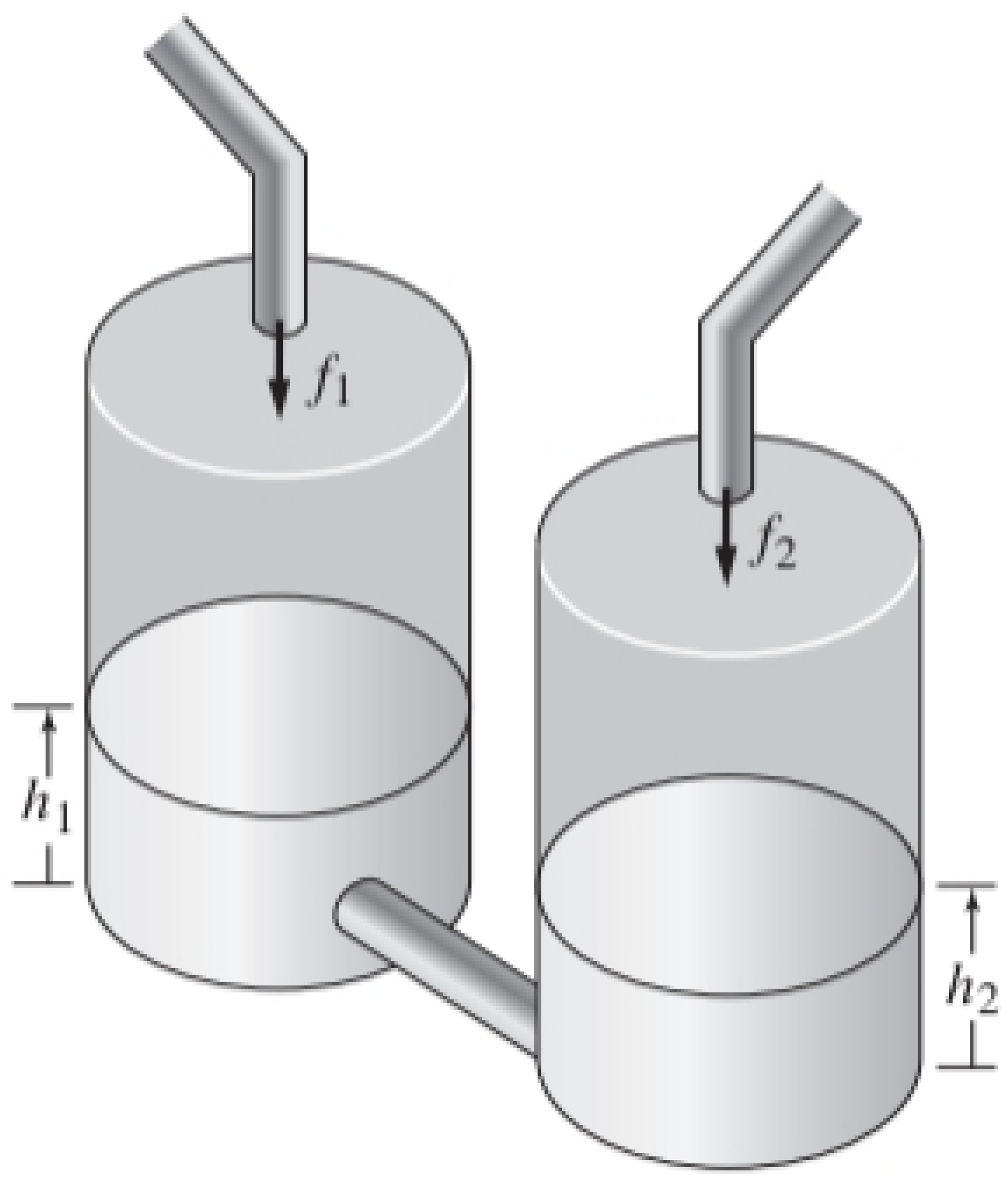



**Figure 1.1.2** Two connected tanks.



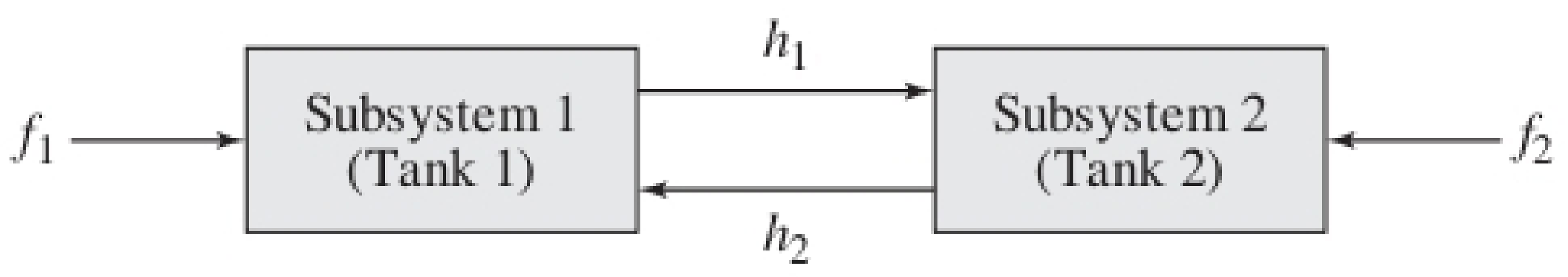



**Figure 1.1.3** A system diagram illustrating how the two liquid heights affect each other.

whose liquid height is influenced by the other tank. We can obtain a differential equation model for each height by using the single-tank relationship $f = bh^m$ and applying the basic physical principle called conservation of mass to express the connection between the two tanks. This results in a model of the entire system.

We often use diagrams to illustrate the connections between the subsystems. Figure 1.1.2 illustrates the physical connection, but Figure 1.1.3 is an example of a diagram showing that the height $h_1$ affects the height $h_2$, and vice versa. (The flow goes from the higher height to the lower one.) Such a diagram may be useful for a nontechnical audience, but it does not show *how* the heights affect each other. To do that, we will use two other types of diagrams—called *simulation* diagrams and *block diagrams*—to represent the connections between the subsystems and the *variables* that describe the system behavior. These diagrams represent the differential equation model.

## 1.1.2 INPUT AND OUTPUT

Like the term "system," the meanings of *input* and *output* have become less precise. For example, a factory manager will call a meeting to seek "input," meaning opinions or data, from the employees, and the manager may refer to the products manufactured in the factory as its "output." However, in the system dynamics meaning of the terms, an *input* is a *cause*; an *output* is an *effect* due to the input. Thus, one input to the bicycle is the force applied to the pedal. One resulting output is the acceleration of the bike. Another input is the angle of the front wheel; the output is the direction of the bike's path of travel.

The behavior of a system element is specified by its *input-output relation*, which is a description of how the output is affected by the input. The input-output relation expresses the cause-and-effect behavior of the element. Such a description, which is represented graphically by the diagram in Figure 1.1.4, can be in the form of a table of numbers, a graph, or a mathematical relation. For example, a force $f$ applied to a particle of mass $m$ causes an acceleration $a$ of the particle. The input-output or causal relation is, from Newton's second law, $a = f/m$. The input is $f$ and the output is $a$.

The input-output relations for the elements in the system provide a means of specifying the connections between the elements. When connected together to form a system, the inputs to some elements will be the outputs from other elements.

The inputs and outputs of a system are determined by the selection of the system's boundary (see Figure 1.1.4). Any causes acting on the system from the world external to this boundary are considered to be system inputs. Similarly, a system's outputs are the outputs from any one or more of the system elements that act on the world outside the system boundary. If we take the bike to be the system, one system input would be the pedal force; another input is the force of gravity acting on the bike. The outputs

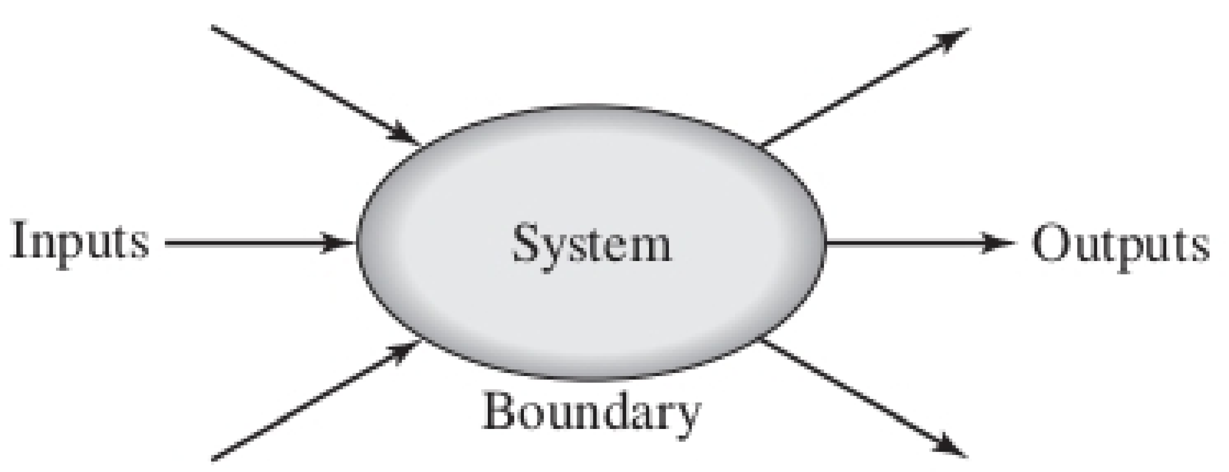
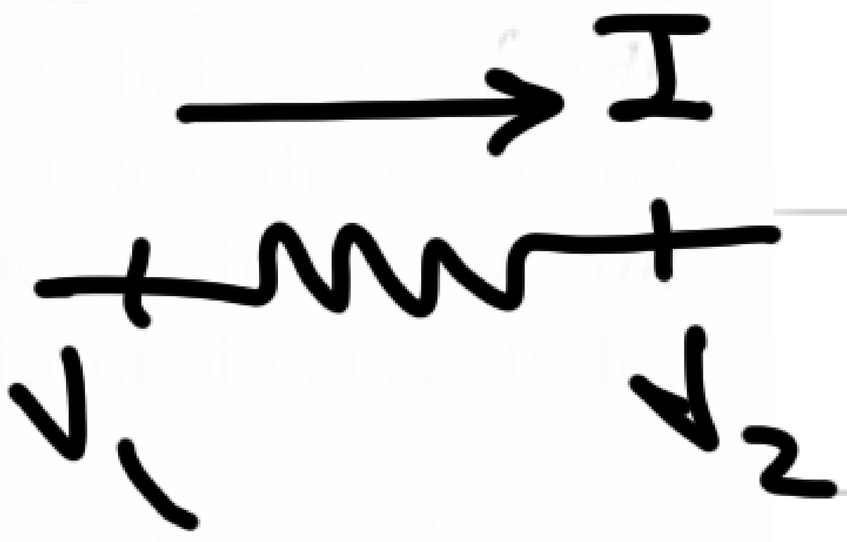



**Figure 1.1.4** A system input-output diagram, showing the system boundary.

may be taken to be the bike's position, velocity, and acceleration. Usually, our choices for system outputs are a subset of the possible outputs and are the variables in which we are interested. For example, a performance analysis of the bike would normally focus on the acceleration or velocity, but not on the bike's position.

Sometimes input-output relations are reversible, sometimes not. For example, we can apply a current as input to a resistor and consider the resulting voltage drop to be the output ($v = iR$). Or we can apply a voltage to produce a current through the resistor ($i = v/R$).

$$V = IR$$

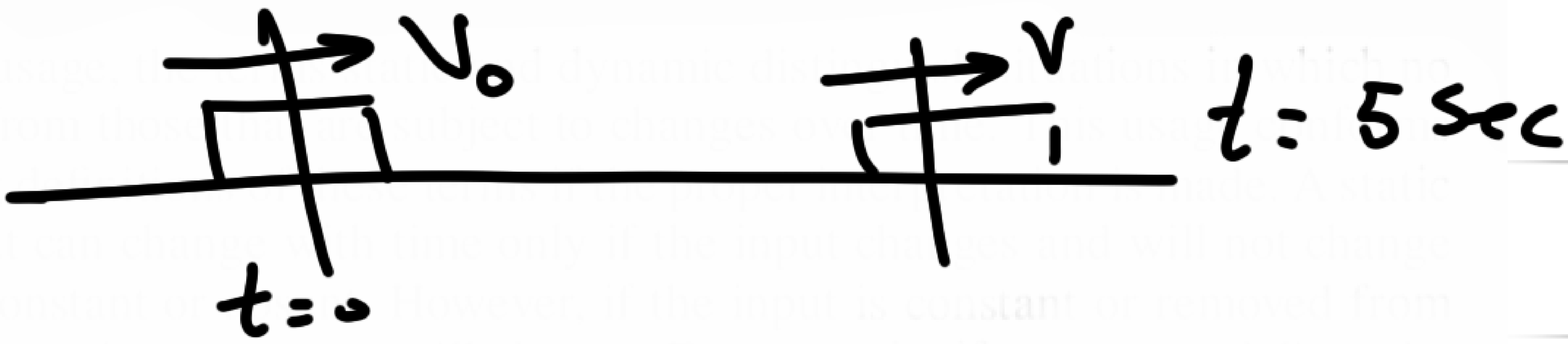



$$\boxed{I = \frac{V}{R}}$$

## 1.1.3 STATIC AND DYNAMIC ELEMENTS

When the present value of an element's output depends only on the present value of its input, we say the element is a *static* element. For example, the current flowing through a resistor depends only on the present value of the applied voltage. The resistor is thus a static element. However, because no physical element can respond instantaneously, the concept of a static element is an approximation. It is widely used, however, because it results in a simpler mathematical representation; that is, an algebraic representation rather than one involving differential equations.

If an element's present output depends on past inputs, we say it is a *dynamic element*. For example, the present position of a bike depends on what its velocity has been from the start.




A *dynamic system* is one whose present output depends on past inputs. A *static system* is one whose output at any given time depends only on the input at that time. A static system contains all static elements. Any system that contains at least one dynamic element must be a dynamic system. *System dynamics*, then, is the study of systems that contain dynamic elements.

## 1.1.4 MODELING OF SYSTEMS

Table 1.1.1 contains a summary of the methodology that has been tried and tested by the engineering profession for many years. These steps describe a general problem-solving procedure. Simplifying the problem sufficiently and applying the appropriate fundamental principles is called *modeling*, and the resulting mathematical description is called a *mathematical model*, or just a *model*. When the modeling has been finished, we need to solve the mathematical model to obtain the required answer. If the model is highly detailed, we may need to solve it with a computer program.

### 1.1.5 MATHEMATICAL METHODS

Because system dynamics deals with changes in time, mathematical models of dynamic systems naturally involve differential equations. Therefore, we introduce differential equation solution methods starting in this chapter. Additional methods, such as those that make use of computers, are introduced in subsequent chapters.

### 1.1.6 CONTROL SYSTEMS

Often dynamic systems require a *control system* to perform properly. Thus, proper control system design is one of the most important objectives of system dynamics. Microprocessors have greatly expanded the applications for control systems. These new applications include robotics, mechatronics, micromachines, precision engineering, active vibration control, active noise cancellation, and adaptive optics. Recent technological advancements mean that many machines now operate at high speeds and high accelerations. It is therefore now more often necessary for engineers to pay more attention to the principles of system dynamics. Starting in Chapter 10, we apply these principles to control system design.

### 1.1.7 APPLICATIONS IN MECHANICAL SYSTEMS

Mechanical systems are loosely defined as those whose operating principles are primarily Newton's laws of motion. The bicycle is an example of a mechanical system. Chapters 3 and 4 deal with mechanical systems. The topic of mechanical vibrations covers the oscillations of machines and structures due either to their own inherent flexibility or to the action of an external force or motion. This is treated in Chapter 13.

Figure 1.1.6 A vehicle suspension system.

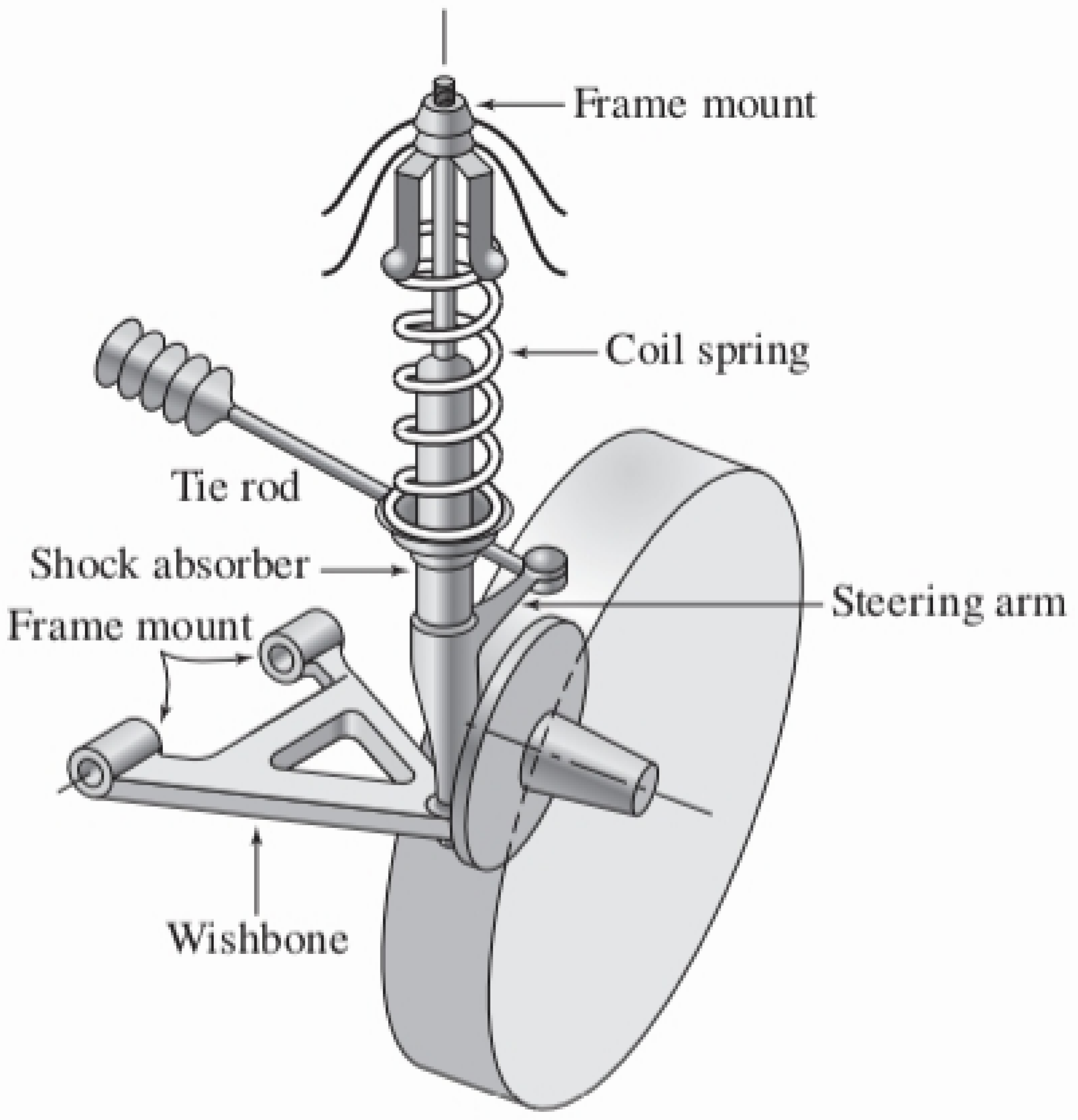
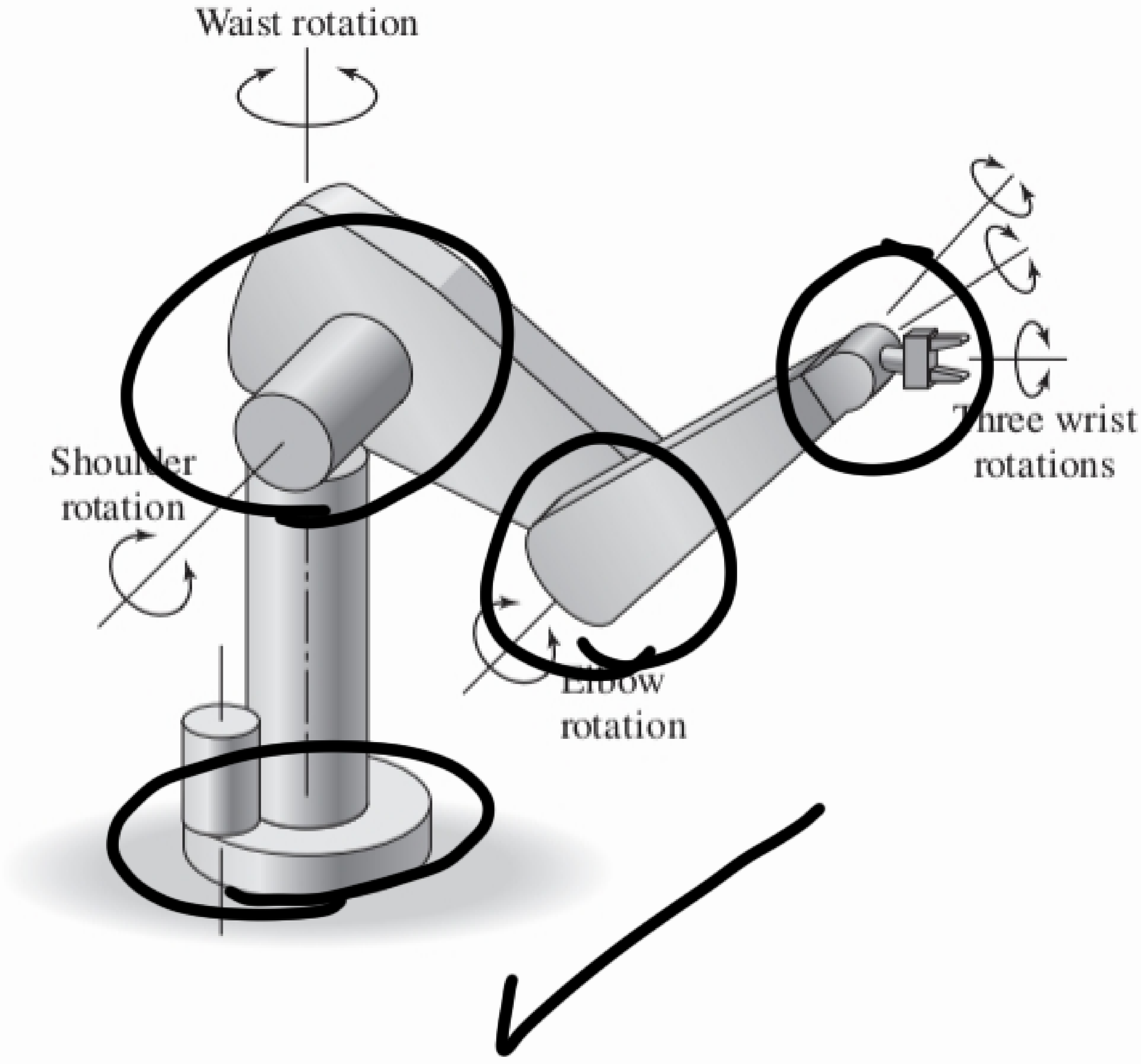



Figure 1.1.7 A robot arm.




One of our major theme applications in mechanical systems is vehicle dynamics. This topic has received renewed importance for reasons related to safety, energy efficiency, and passenger comfort. Of major interest under this topic is the design of vehicle suspension systems, whose elements include various types of springs and shock absorbers (Figure 1.1.6). *Active* suspension systems, whose characteristics can be changed under computer control, and vehicle-dynamics control systems are undergoing rapid development, and their design requires an understanding of system dynamics.

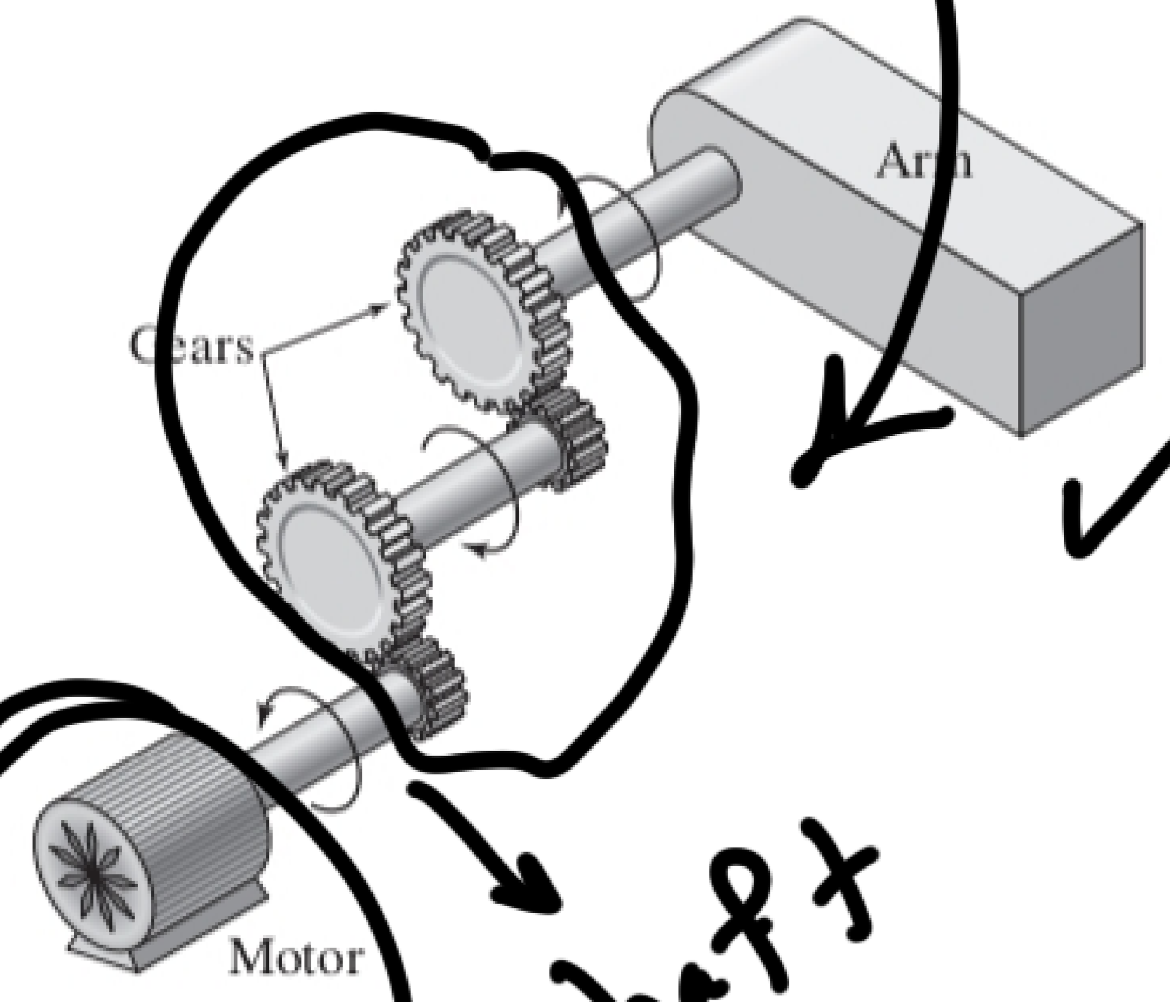
### 1.1.8 APPLICATIONS IN ELECTRICAL AND ELECTROMECHANICAL SYSTEMS

Electromechanical systems contain both mechanical elements and electrical elements such as electric motors. Two common applications of system dynamics in electromechanical systems are in (1) motion-control systems and (2) vehicle dynamics. Therefore, we will use these applications as major themes in many of our examples and problems. Chapter 6 introduces electrical and electromechanical systems.

Figure 1.1.7 shows a robot arm, whose motion must be properly controlled to move an object to a desired position and orientation. To do this, each of the several motors and drive trains in the arm must be adequately designed to handle the load, and the motor speeds and angular positions must be properly controlled. Figure 1.1.8 shows a typical motor and drive train for one arm joint. Knowledge of system dynamics is essential to design these subsystems and to control them properly.

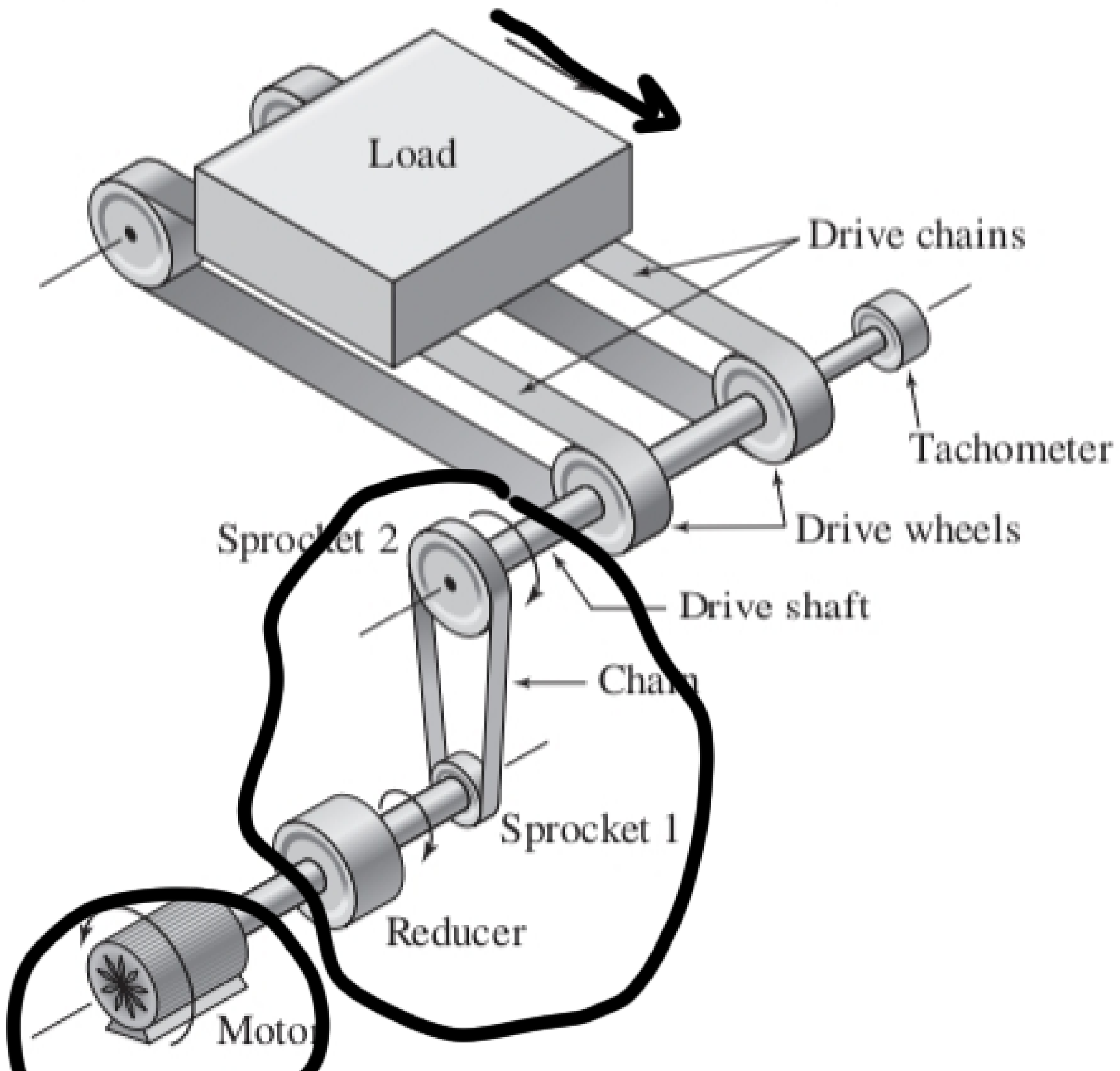
Mobile robots are another motion-control application, but motion-control applications are not limited to robots. Figure 1.1.9 shows the mechanical drive for a conveyor system. The motor, the gears in the speed reducer, the chain, the sprockets, and the drive wheels all must be properly selected, and the motor must be properly controlled for the system to work well. In subsequent chapters we will develop models of these components and use them to design the system and analyze its performance.

Figure 1.1.8 Mechanical drive for a robot arm joint.




shaft

Figure 1.1.9 Mechanical drive for a conveyor system.




## 1.1.9 APPLICATIONS IN FLUID SYSTEMS

A fluid system is one whose operation depends on the flow of a fluid. If the fluid is *incompressible*, that is, if its density does not change appreciably with pressure changes, we call it a liquid, or a *hydraulic* fluid. On the other hand, if the fluid is *compressible*, that is, if its density does change appreciably with pressure changes, we call it a gas, or a *pneumatic* fluid.

Figure 1.1.10 shows a commonly seen backhoe. The bucket, forearm, and upper arm are each driven by a *hydraulic servomotor*. A cutaway view of such a motor is shown in Figure 1.1.11. We will analyze its behavior in Chapter 7. Compressed air

Figure 1.1.10 A backhoe.

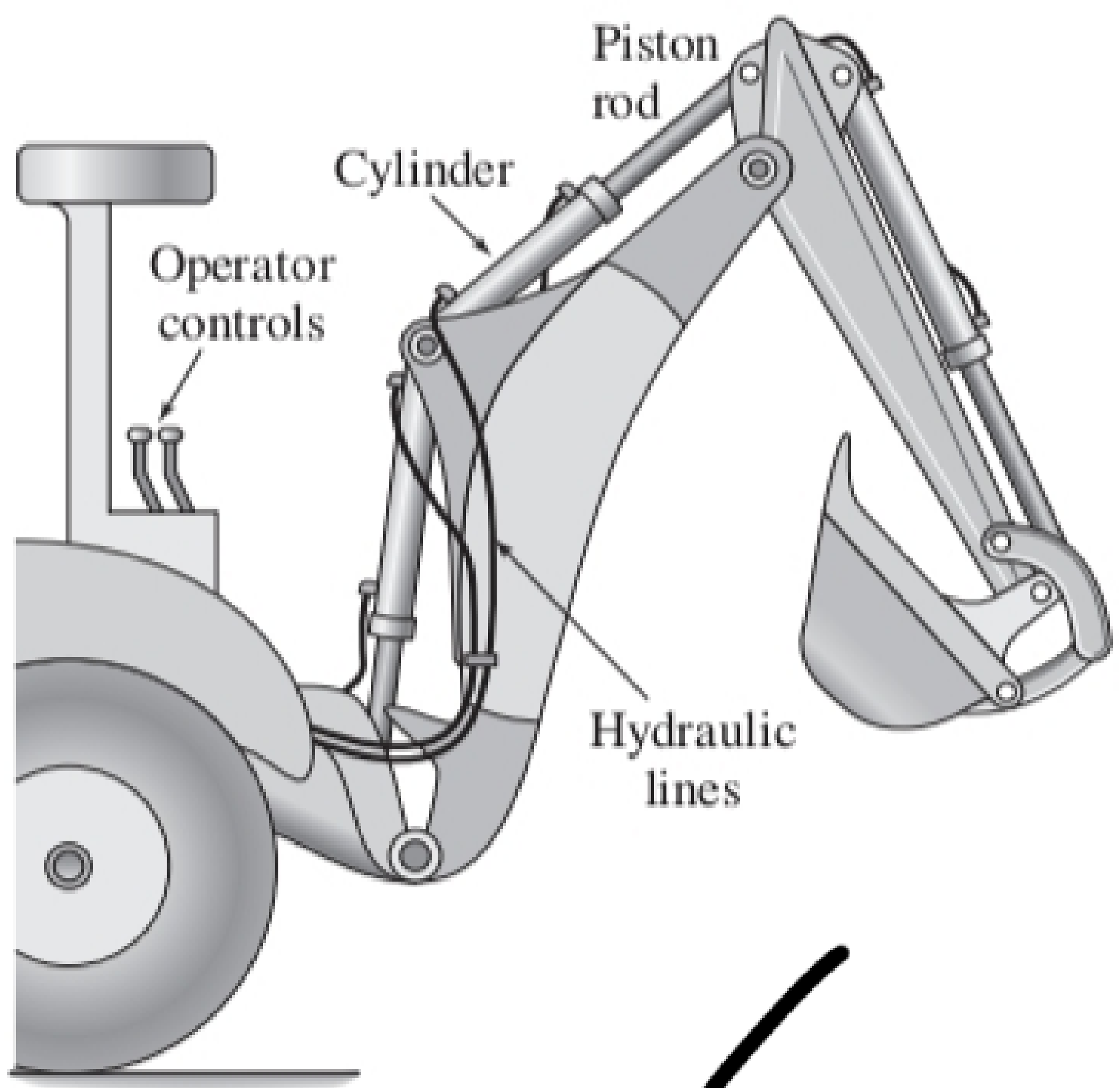
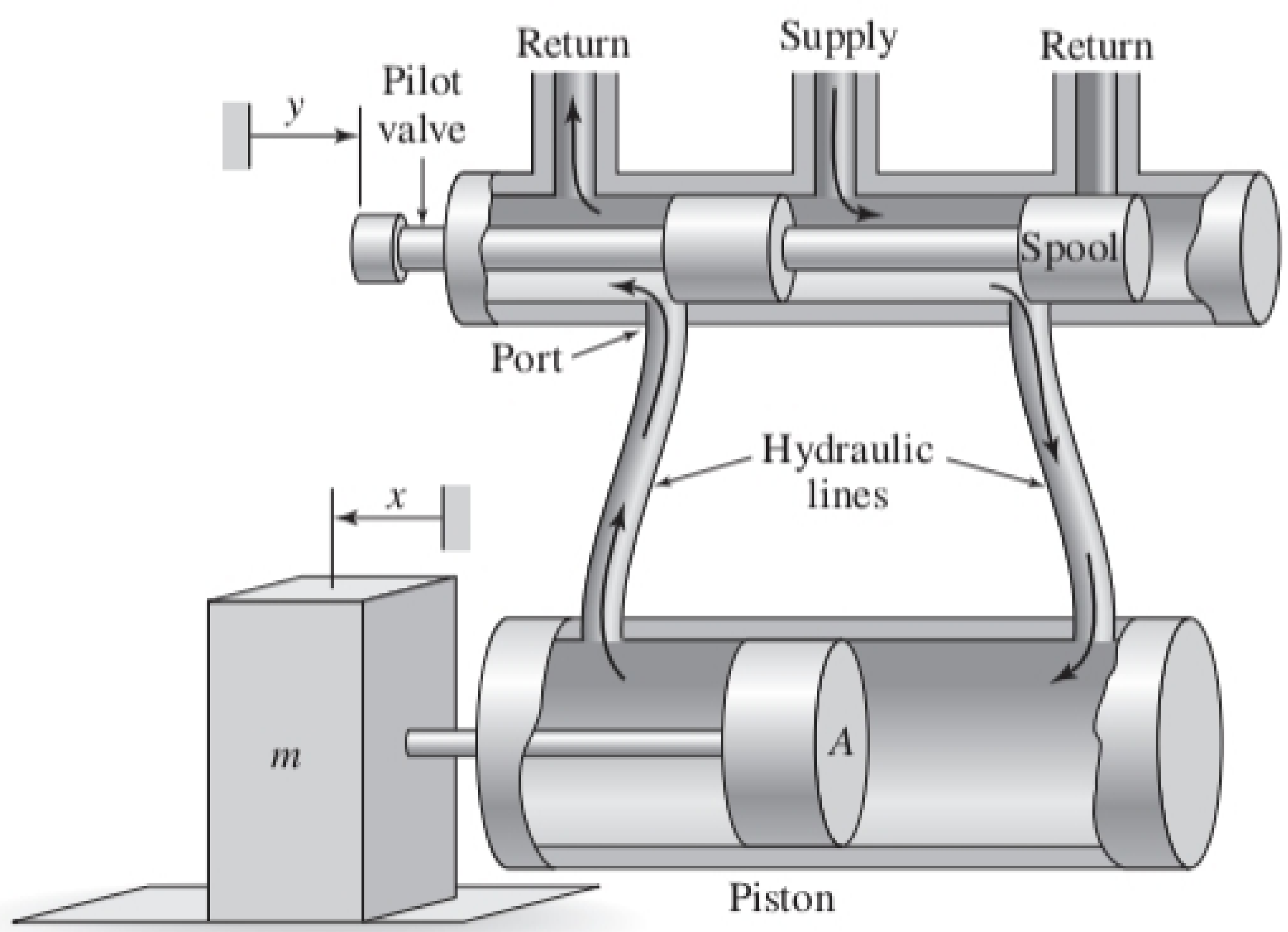



Figure 1.1.11 A hydraulic servomotor.



### 1.1.10 APPLICATIONS IN THERMAL SYSTEMS

A thermal system is one whose behavior depends primarily on the exchange of heat. The oven-potato application we saw earlier is an example of a thermal system. Many thermal systems involve fluid flow, such as with a steam engine or an air conditioner, and so we often speak of *thermo-fluid systems*. These examples also have mechanical components such as pistons, and so we could refer to them as thermo-fluid-mechanical systems, although we rarely use such cumbersome terminology. The designation as thermal, fluid, or mechanical depends on what aspect of the system we are analyzing. Thermal systems are first treated in Chapter 7, with more applications covered in later chapters.

### 1.1.11 COMPUTER METHODS

The computer methods used in this text are based on MATLAB and Simulink.®[1] If you are unfamiliar with MATLAB, Appendix D on the textbook website contains a thorough introduction to the program. No prior experience with Simulink is required; we will introduce the necessary methods as we need them. For the convenience of those who prefer to use a software package other than MATLAB or Simulink, we have placed all the MATLAB and Simulink material in optional sections at the end of each chapter. They can be skipped without affecting your understanding of the following chapters. If you use a program, such as MATLAB, to solve a problem, follow the steps shown in Table 1.1.2.

## 1.2 UNITS

In this book we use two systems of units, the *FPS* system and the metric SI. The common system of units in business and industry in English-speaking countries has been the foot-pound-second (FPS) system. This system is also known as the U.S. customary system or the British Engineering system. Much engineering work in the United States has been based on the FPS system, and some industries continue to use it. The metric Système International d'Unités (SI) nevertheless is becoming the worldwide standard.

SI unit & FPS

Foot Pound second.

الوحدات الإنجليزية

او الأمريكية

Until the changeover is complete, engineers in the United States will have to be familiar with both systems.

In our examples, we will use SI and FPS units in the hope that the student will become comfortable with both. Other systems are in use, such as the meter-kilogram-second (mks) and centimeter-gram-second (cgs) metric systems and the British system, in which the mass unit is a pound. We will not use these, in order to simplify our coverage and because FPS and SI units are the most common in engineering applications. We now briefly summarize these two systems.

### 1.2.1 FPS UNITS

The FPS system is a *gravitational* system. This means that the primary variable is force, and the unit of mass is derived from Newton's second law. The *pound* is selected as the unit of force and the *foot* and *second* as units of length and time, respectively. From Newton's second law of motion, force equals mass times acceleration, or

$$f = ma \tag{1.2.1}$$

where $f$ is the net force acting on the mass $m$ and producing an acceleration $a$. Thus, the unit of mass must be

$$\text{mass} = \frac{\text{force}}{\text{acceleration}} = \frac{\text{pound}}{\text{foot/(second)}^2}$$

This mass unit is named the *slug*.

Through Newton's second law, the weight $W$ of an object is related to the object mass $m$ and the acceleration due to gravity, denoted by $g$, as follows: $W = mg$. At the surface of the earth, the standard value of $g$ in FPS units is $g = 32.2\ \text{ft/sec}^2$.

Energy has the dimensions of mechanical work; namely, force times displacement. Therefore, the unit of energy in this system is the *foot-pound* (ft-lb). Another energy unit in common use for historical reasons is the *British thermal unit* (Btu). The relationship between the two is given in Table 1.2.1. Power is the rate of change of energy with time, and a common unit is *horsepower*. Finally, temperature in the FPS system can be expressed in degrees *Fahrenheit* or in absolute units, degrees *Rankine*.

### 1.2.2 SI UNITS

The SI metric system is an *absolute* system, which means that the mass is chosen as the primary variable, and the force unit is derived from Newton's law. The *meter* and the

**Table 1.2.1** SI and FPS units.

| | Unit name and abbreviation | |
|---|---|---|
| **Quantity** | **SI Unit** | **FPS Unit** |
| Time | second (s) | second (sec) |
| Length | meter (m) | foot (ft) |
| Force | newton (N) | pound (lb) |
| Mass | kilogram (kg) | slug |
| Energy | joule (J) | foot-pound (ft-lb), Btu (= 778 ft-lb) |
| Power | watt (W) | ft-lb/sec, horsepower (hp) |
| Temperature | degrees Celsius (°C), degrees Kelvin (K) | degrees Fahrenheit (°F), degrees Rankine (°R) |

**Table 1.2.2** Unit conversion factors.

| | | |
|---|---|---|
| Length | 1 m = 3.281 ft | 1 ft = 0.3048 m |
| | 1 mile = 5280 ft | 1 km = 1000 m |
| Speed | 1 ft/sec = 0.6818 mi/hr | 1 mi/hr = 1.467 ft/sec |
| | 1 m/s = 3.6 km/h | 1 km/h = 0.2778 m/s |
| | 1 km/hr = 0.6214 mi/hr | 1 mi/hr = 1.609 km/h |
| Force | 1 N = 0.2248 lb | 1 lb = 4.4484 N |
| Mass | 1 kg = 0.06852 slug | 1 slug = 14.594 kg |
| Energy | 1 J = 0.7376 ft-lb | 1 ft-lb = 1.3557 J |
| Power | 1 hp = 550 ft-lb/sec | 1 hp = 745.7 W |
| | 1 W = $1.341 \times 10^{-3}$ hp | |
| Temperature | $T$°C = 5 ($T$°F − 32)/9 | $T$°F = 9$T$°C/5 + 32 |

*second* are selected as the length and time units, and the *kilogram* is chosen as the mass unit. The derived force unit is called the *newton*. In SI units the common energy unit is the newton-meter, also called the *joule*, while the power unit is the joule/second, or *watt*. Temperatures are measured in degrees Celsius, °C, and in absolute units, which are degrees *Kelvin*, K. The difference between the boiling and freezing temperatures of water is 100°C, with 0°C being the freezing point.

At the surface of the earth, the standard value of $g$ in SI units is $g = 9.81$ m/s$^2$.

Table 1.2.2 gives the most commonly needed factors for converting between the FPS and the SI systems.

ω = frequency = rad/sec

### 1.2.3 OSCILLATION UNITS

There are three commonly used units for frequency of oscillation. If time is measured in seconds, frequency can be specified as *radians/second* or as *hertz*, abbreviated Hz. One hertz is one cycle per second (cps). The relation between cycles per second $f$ and radians per second $\omega$ is $2\pi f = \omega$. For sinusoidal oscillation, the *period* $P$, which is the time between peaks, is related to frequency by $P = 1/f = 2\pi/\omega$. The third way of specifying frequency is revolutions per minute (rpm). Because there are $2\pi$ radians per revolution, one rpm = $(2\pi/60)$ radians per second.

$\omega = 2\pi f$

$\boxed{f = \frac{\omega}{2\pi}}$

$f = \sim$ Hz

$N \rightarrow$ r.p.m

$\boxed{\omega = \frac{2\pi N}{60}}$

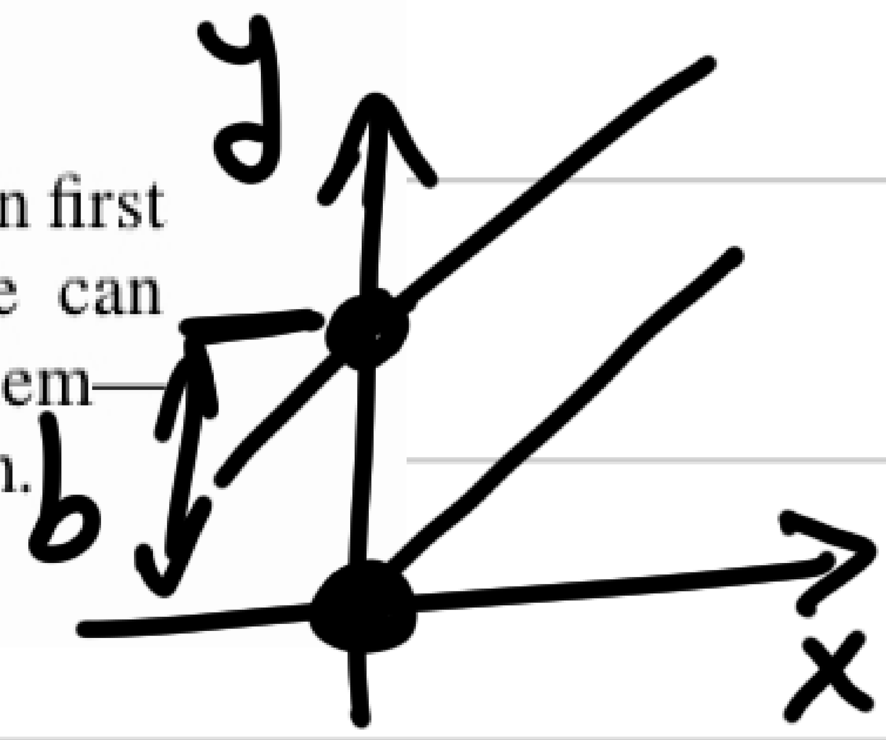
$\boxed{T = \frac{1}{f}}$ (sec)

## 1.3 DEVELOPING LINEAR MODELS

A *linear* model of a static element has the form $y = mx + b$, where $x$ is the input and $y$ is the output of the element. As we will see in Chapter 2, solution of dynamic models to predict system performance requires solution of differential equations. Differential equations based on linear models of the system elements are easier to solve than ones based on nonlinear models. Therefore, when developing models we try to obtain a linear model whenever possible. Sometimes the use of a linear model results in a loss of accuracy, and the engineer must weigh this disadvantage with advantages gained by using a linear model. In this section, we illustrate some ways to obtain linear models.

### 1.3.1 DEVELOPING LINEAR MODELS FROM DATA

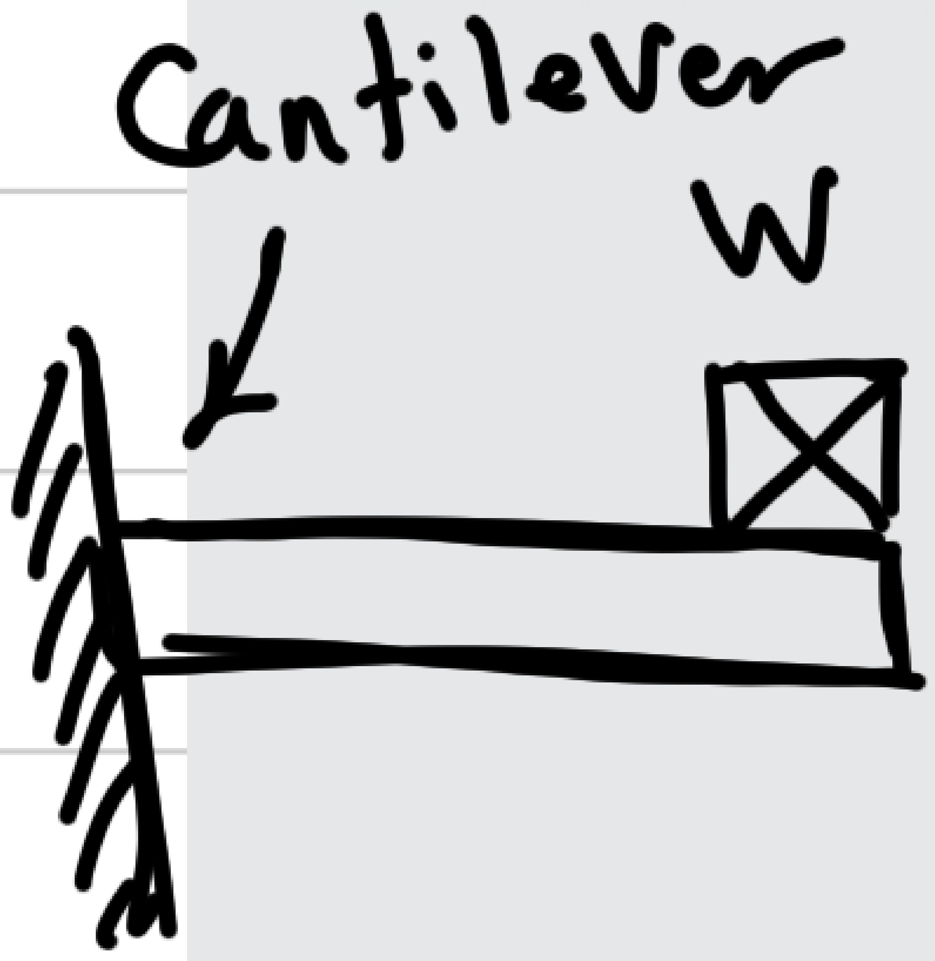
If we are given data on the input-output characteristics of a system element, we can first plot the data to see whether a linear model is appropriate, and if so, we can extract a suitable model. Example 1.3.1 illustrates a common engineering problem—the estimation of the force-deflection characteristics of a cantilever support beam.




$\boxed{y = mx + b}$

**EXAMPLE 1.3.1**

## A Cantilever Beam Deflection Model




**■ Problem**

The deflection of a cantilever beam is the distance its end moves in response to a force applied at the end (Figure 1.3.1). This distance is called the *deflection* and it is the output variable. The applied force is the input. The following table gives the measured deflection $x$ that was produced in a particular beam by the given applied force $f$. Plot the data to see whether a linear relation exists between $f$ and $x$.

| x ✓ Force $f$ (lb) | 0 | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
|---|---|---|---|---|---|---|---|---|---|
| y ✓ Deflection $x$ (in.) | 0 | 0.15 | 0.23 | 0.35 | 0.37 | 0.5 | 0.57 | 0.68 | 0.77 |

**■ Solution**

The plot is shown in Figure 1.3.2. Common sense tells us that there must be zero beam deflection if there is no applied force, so the curve describing the data must pass through the origin. The straight line shown was drawn by aligning a straightedge so that it passes through the origin

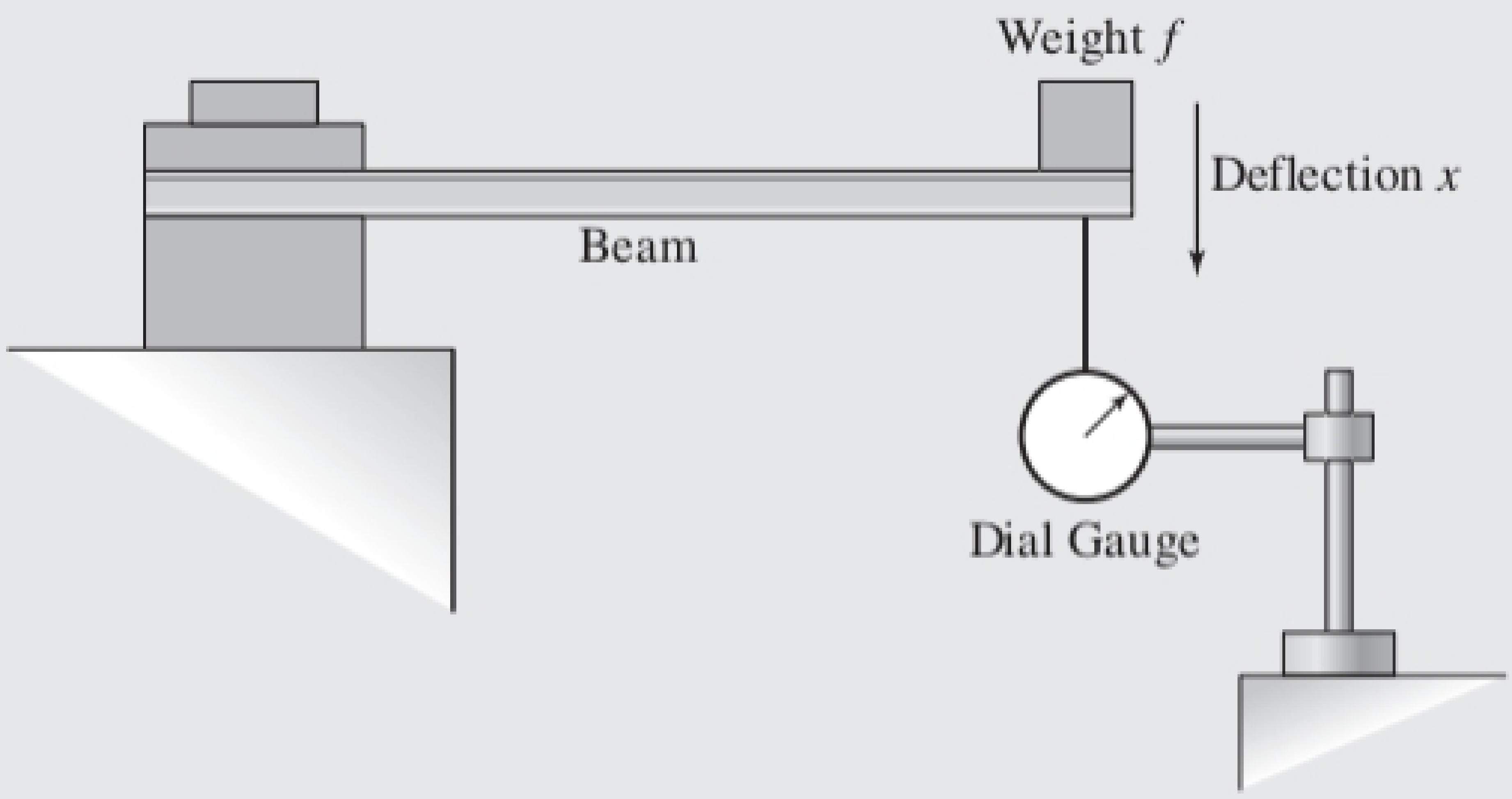



**Figure 1.3.1** Measurement of beam deflection.

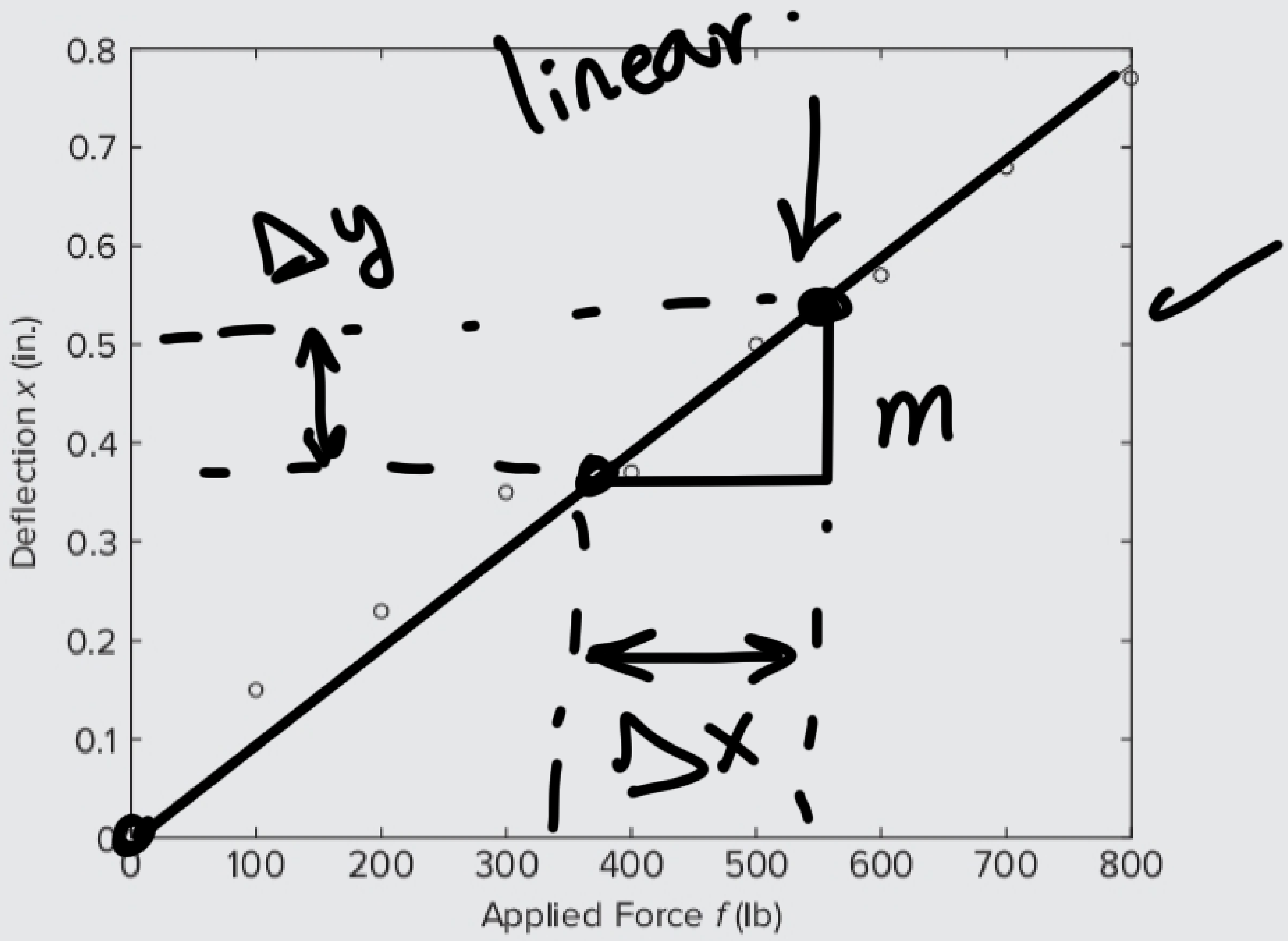



**Figure 1.3.2** Plot of beam deflection versus applied force.

y

$y = mx + \cancel{b}$ 0

x

$m = \frac{\Delta y}{\Delta x}$

slope

$\frac{\Delta y}{\Delta f} = m$

$k = \frac{1}{m}$

$K = \frac{f}{x}$ $\boxed{f = k x}$

and near most of the data points (note that this line is subjective; another person might draw a different line). The data lie close to a straight line, so we can use the linear function $x = af$ to describe the relation. The value of the constant $a$ can be determined from the slope of the line. Choosing the origin and the last data point to find the slope, we obtain

$$a = \frac{0.78 - 0}{800 - 0} = 9.75 \times 10^{-4} \text{ in./lb}$$

As we will see in Chapter 4, this relation is usually written as $f = kx$, where $k$ is called the beam *stiffness*. Thus, $k = 1/a = 1025$ lb/in.

Once we have discovered a functional relation that describes the data, we can use it to make predictions for conditions that lie *within* the range of the original data. This process is called *interpolation*. For example, we can use the beam model to estimate the deflection when the applied force is 550 lb. We can be fairly confident of this prediction because we have data below and above 550 lb and we have seen that our model describes these data very well.

*Extrapolation* is the process of using the model to make predictions for conditions that lie *outside* the original data range. Extrapolation might be used in the beam application to predict how much force would be required to bend the beam 1.2 in. We must be careful when using extrapolation, because we usually have no reason to believe that the mathematical model is valid beyond the range of the original data. For example, if we continue to bend the beam, eventually the force is no longer proportional to the deflection, and it becomes much greater than that predicted by the linear model. Extrapolation has a use in making tentative predictions, which must be backed up later on by testing.

In some applications, the data contain so much scatter that it is difficult to identify an appropriate straight line. In such cases, we must resort to a more systematic and objective way of obtaining a model. This topic is treated in Appendix C.

## 1.3.2 LINEARIZATION

Not all element descriptions are in the form of data. Often we know the analytical form of the model, and if the model is nonlinear, we can obtain a linear model that is an accurate approximation over a limited range of the independent variable. Examples 1.3.2 and 1.3.3 illustrate this technique, which is called *linearization*.

### Linearization of the Sine Function — EXAMPLE 1.3.2

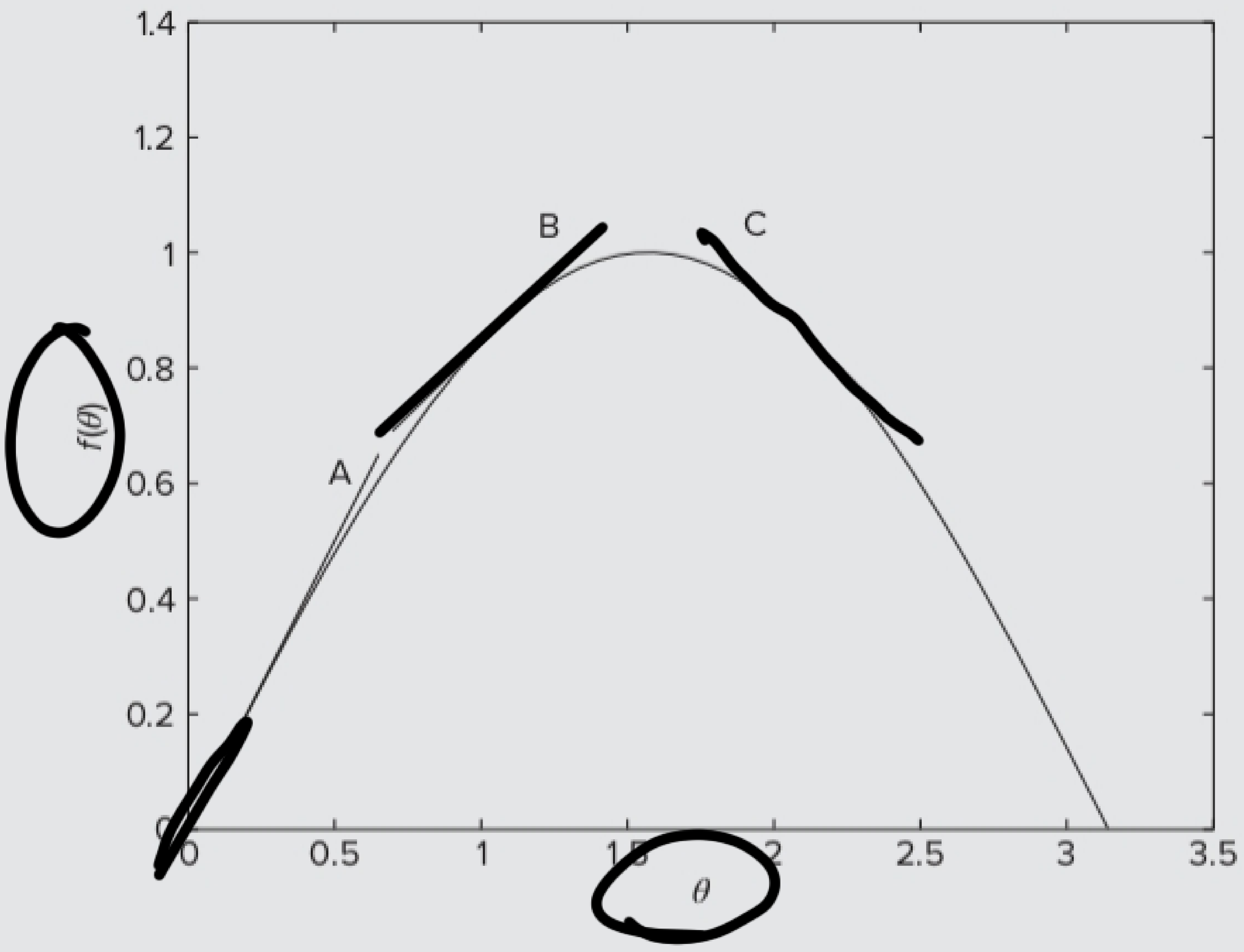
**■ Problem**

We will see in Chapter 3 that the models of many mechanical systems involve the sine function $\sin\theta$, which is nonlinear. Obtain three linear approximations of $f(\theta) = \sin\theta$, one valid near $\theta = 0$, one near $\theta = \pi/3$ rad (60°), and one near $\theta = 2\pi/3$ rad (120°).

**■ Solution**

The essence of the linearization technique is to replace the plot of the nonlinear function with a straight line that passes through the reference point and has the same slope as the nonlinear function at that point. Figure 1.3.3 shows the sine function and the three straight lines obtained with this technique. Note that the slope of the sine function is its derivative, $d \sin\theta/d\theta = \cos\theta$, and thus the slope is not constant but varies with $\theta$.

Figure 1.3.3 Three linearized models of the sine function.




Consider the first reference point, $\theta = 0$. At this point the sine function has the value $\sin 0 = 0$, the slope is $\cos 0 = 1$, and thus the straight line passing through this point with a slope of 1 is $f(\theta) = \theta$. This is the linear approximation of $f(\theta) = \sin\theta$ valid near $\theta = 0$, line A in Figure 1.3.3. Thus, we have derived the commonly seen small-angle approximation $\sin\theta \approx \theta$.

Next consider the second reference point, $\theta = \pi/3$ rad. At this point the sine function has the value $\sin \pi/3 = 0.866$, the slope is $\cos \pi/3 = 0.5$, and thus the straight line passing through this point with a slope of 0.5 is $f(\theta) = 0.5(\theta - \pi/3) + 0.866$, line B in Figure 1.3.3. This is the linear approximation of $f(\theta) = \sin\theta$ valid near $\theta = \pi/3$.

Now consider the third reference point, $\theta = 2\pi/3$ rad. At this point the sine function has the value $\sin 2\pi/3 = 0.866$, the slope is $\cos 2\pi/3 = -0.5$, and thus the straight line passing through this point with a slope of $-0.5$ is $f(\theta) = -0.5(\theta - 2\pi/3) + 0.866$, line C in Figure 1.3.3. This is the linear approximation of $f(\theta) = \sin\theta$ valid near $\theta = 2\pi/3$.

In Example 1.3.2 we used a graphical approach to develop the linear approximation. The linear approximation can also be developed with an analytical approach based on the Taylor series. The Taylor series represents a function $f(\theta)$ in the vicinity of $\theta = \theta_r$ by the expansion

$$f(\theta) = f(\theta_r) + \left(\frac{df}{d\theta}\right)_{\theta=\theta_r}(\theta - \theta_r) + \frac{1}{2}\left(\frac{d^2f}{d\theta^2}\right)_{\theta=\theta_r}(\theta - \theta_r)^2 + \cdots$$
$$+ \frac{1}{k!}\left(\frac{d^kf}{d\theta^k}\right)_{\theta=\theta_r}(\theta - \theta_r)^k + \cdots \tag{1.3.1}$$

Consider the nonlinear function $f(\theta)$, which is sketched in Figure 1.3.4. Let $[\theta_r, f(\theta_r)]$ denote the reference operating condition of the system. A model that is linear can be obtained by expanding $f(\theta)$ in a Taylor series near this point and truncating

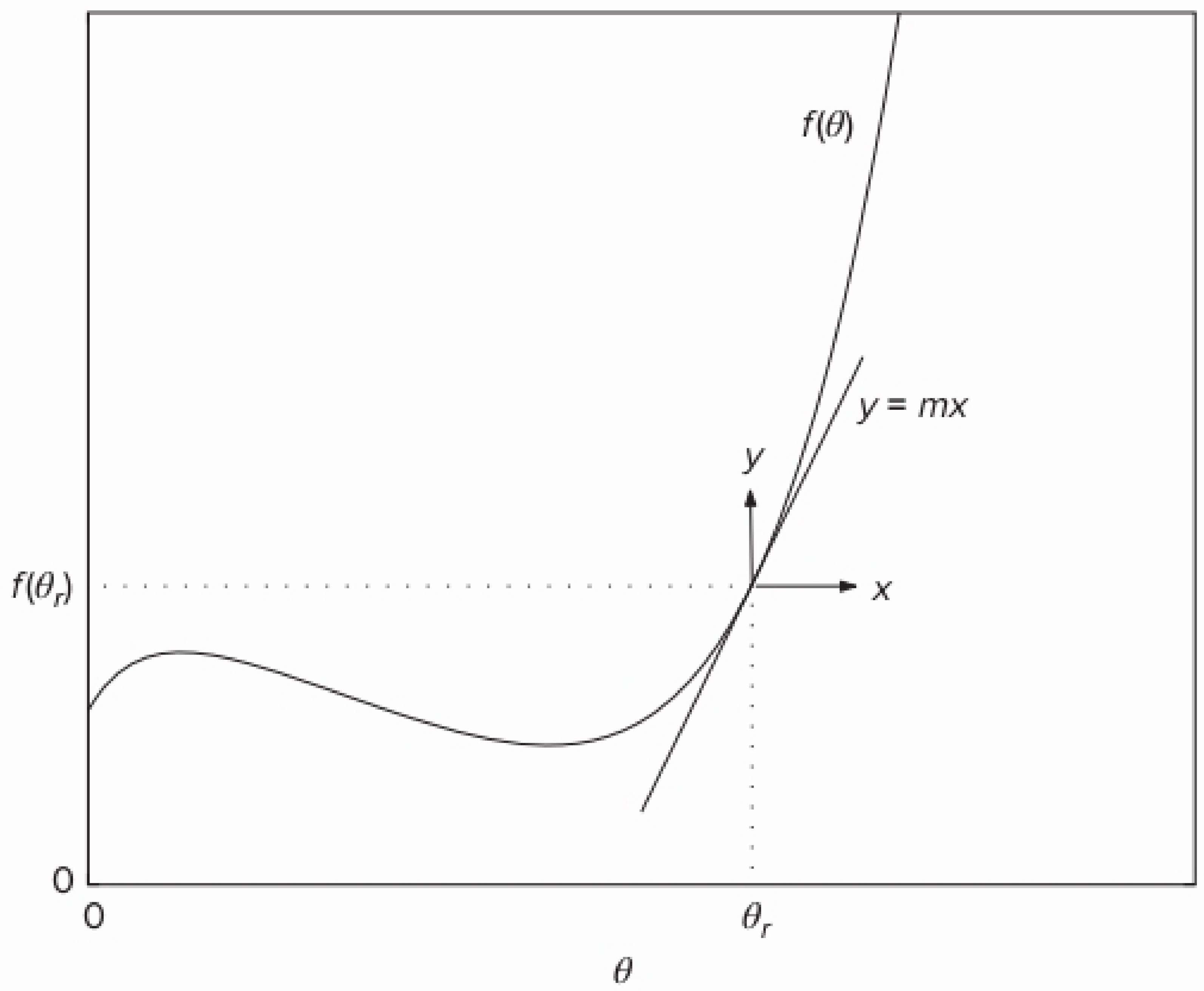



**Figure 1.3.4** Graphical interpretation of function linearization.

the series beyond the first-order term. If $\theta$ is "close enough" to $\theta_r$, the terms involving $(\theta - \theta_r)^i$ for $i \geq 2$ are small compared to the first two terms in the series. The result is

$$f(\theta) = f(\theta_r) + \left(\frac{df}{d\theta}\right)_r (\theta - \theta_r) \tag{1.3.2}$$

where the subscript $r$ on the derivative means that it is evaluated at the reference point. This is a linear relation. To put it into a simpler form, let $m$ denote the slope at the reference point.

$$m = \left(\frac{df}{d\theta}\right)_r \tag{1.3.3}$$

Let $y$ denote the difference between $f(\theta)$ and the reference value $f(\theta_r)$.

$$y = f(\theta) - f(\theta_r) \tag{1.3.4}$$

Let $x$ denote the difference between $\theta$ and the reference value $\theta_r$.

$$x = \theta - \theta_r \tag{1.3.5}$$

Then (1.3.2) becomes

$$y = mx \tag{1.3.6}$$

The geometric interpretation of this result is shown in Figure 1.3.4. We have replaced the original function $f(\theta)$ with a straight line passing through the point $[\theta_r, f(\theta_r)]$ and having a slope equal to the slope of $f(\theta)$ at the reference point. Using the $(y, x)$ coordinates gives a zero intercept, and simplifies the relation.

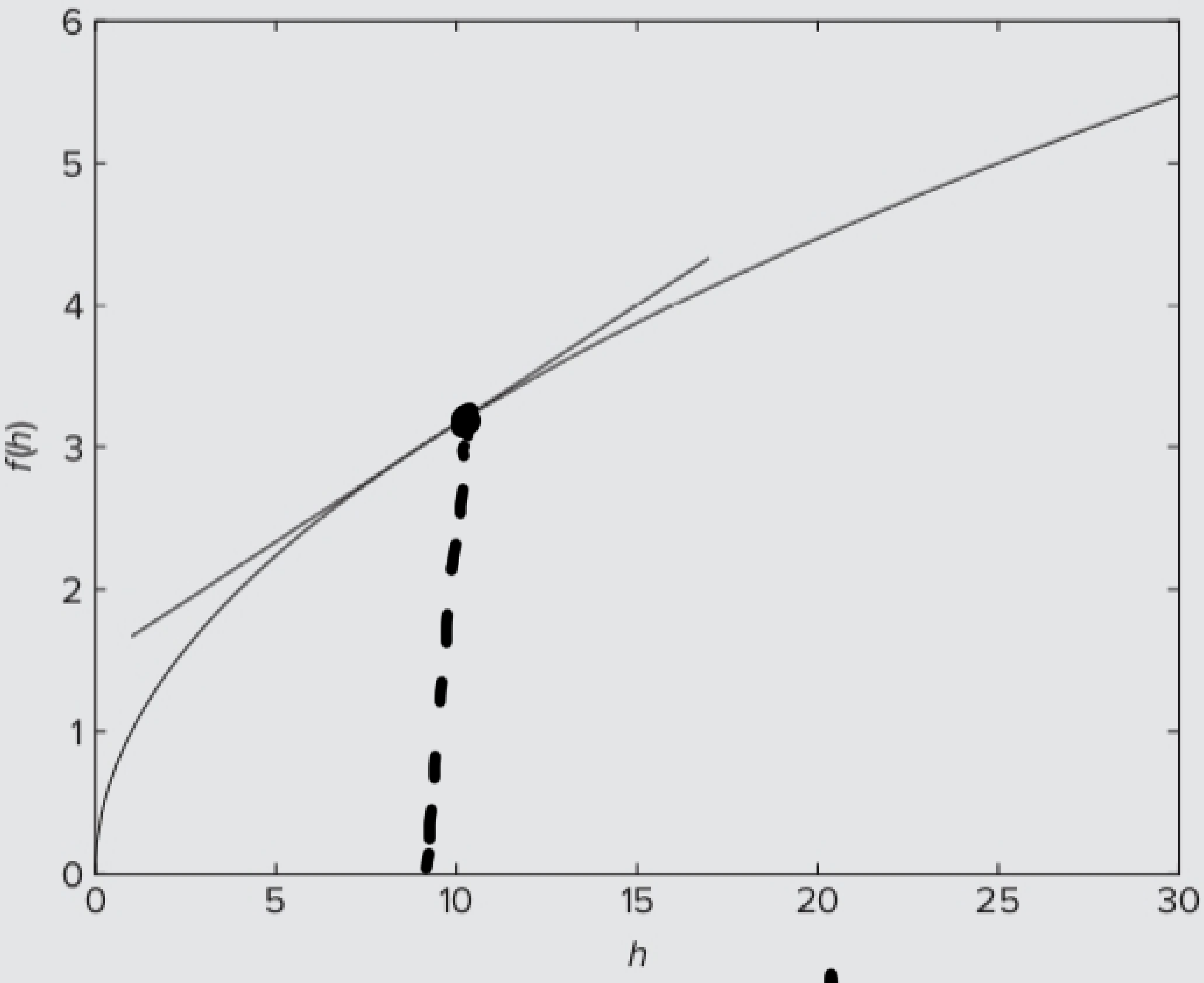
## Linearization of a Square-Root Model

**EXAMPLE 1.3.3**

**■ Problem**

We will see in Chapter 7 that the models of many fluid systems involve the square-root function $\sqrt{h}$, which is nonlinear. Obtain a linear approximation of $f(h) = \sqrt{h}$ valid near $h = 9$.

**Figure 1.3.5** Linearization of the square-root function.




$\frac{d}{dh}\left[h^{1/2}\right]$

$\frac{1}{2} h^{-1/2}$

■ **Solution**

The truncated Taylor series for this function is

$$\boxed{f(h) = f(h_r) + \left.\frac{d\left(\sqrt{h}\right)}{dh}\right|_r (h - h_r)}$$

where $h_r = 9$. This gives the linear approximation

$$\boxed{f(h) = \sqrt{9} + \left.\frac{1}{2}h^{-1/2}\right|_r (h - 9) = 3 + \frac{1}{6}(h - 9)}$$

This equation gives the straight line shown in Figure 1.3.5.

Sometimes we need a linear model that is valid over so wide a range of the independent variable that a model obtained from the Taylor series is inaccurate or grossly incorrect. In such cases, we must settle for a linear function that gives a conservative estimate.

**EXAMPLE 1.3.4**

## Modeling Fluid Drag

$D \propto v^2$

■ **Problem**

The drag force on an object moving through a liquid or a gas is a function of the velocity. A commonly used model of the drag force $D$ on an object is

$$D = \frac{1}{2}\rho A C_D v^2 \tag{1}$$

where $\rho$ is the mass density of the fluid, $A$ is the object's cross-sectional area normal to the relative flow, $v$ is the object's velocity relative to the fluid, and $C_D$ is the *drag coefficient*, which is usually determined from wind-tunnel or water-channel tests on models. Curve A in Figure 1.3.6 is a plot of this equation for an Aerobee rocket 1.25 ft in diameter, with $C_D = 0.4$, moving through the lower atmosphere where $\rho = 0.0023$ slug/ft$^3$, for which equation (1) becomes

$$D = 0.00056v^2 \tag{2}$$

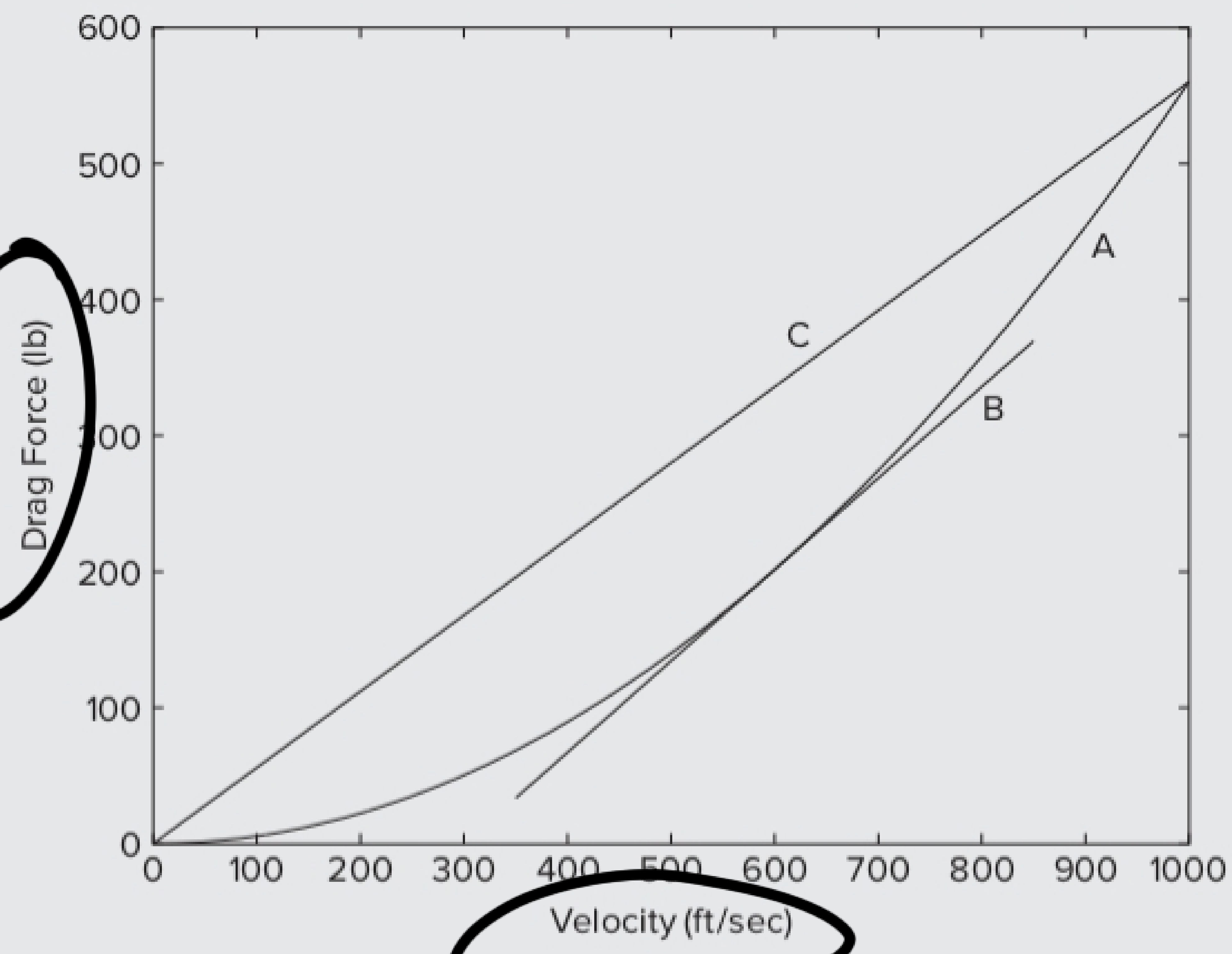



**Figure 1.3.6** Models of fluid drag.

a. Obtain a linear approximation to this drag function valid near $v = 600$ ft/sec.
b. Obtain a linear approximation that gives a conservative (high) estimate of the drag force as a function of the velocity over the range $0 \le v \le 1000$ ft/sec.

■ **Solution**

a. The Taylor series approximation of equation (2) near $v = 600$ is

$$D = D|_{v=600} + \left.\frac{dD}{dv}\right|_{v=600} (v - 600) = 201.6 + 0.672(v - 600)$$

This straight line is labeled B in Figure 1.3.6. Note that it predicts that the drag force will be negative when the velocity is less than 300 ft/sec, a result that is obviously incorrect. This illustrates how we must be careful when using linear approximations.

b. The linear model that gives a conservative estimate of the drag force (that is, an estimate that is never less than the actual drag force) is the straight-line model that passes through the origin and the point at $v = 1000$. This is the equation $D = 0.56v$, shown by the straight line C in Figure 1.3.6.

## 1.4 INTRODUCTION TO DIFFERENTIAL EQUATIONS

A *differential equation* is simply an equation involving derivatives with respect to the independent variable. In system dynamics this variable is often the time variable, usually denoted by $t$. This section considers some simple problems in which the derivative can be isolated and the basic methods of integral calculus can be applied to obtain a solution. We also discuss the role of the *initial conditions* in obtaining a solution. The approach is illustrated with a simple dynamics application: the challenge of bringing a planetary lander to a soft landing. Time derivatives are often abbreviated with the dot notation, as

$$\dot{x} = \frac{dx}{dt} \qquad \ddot{x} = \frac{d^2x}{dt^2}$$

$x \rightarrow$ displacement

$\dot{x} \rightarrow$ velocity

$\ddot{x} \rightarrow$ acceleration

$\frac{dx}{dt} = \dot{x}$

$\frac{dx^2}{dt^2} = \ddot{x}$

### 1.4.1 SOLUTION BY DIRECT INTEGRATION

With some equations involving only a first-order derivative, sometimes we can isolate the derivative on the left, as

$x @ t = t_f$

$$\frac{dx}{dt} = f(t)$$

$\int dx = x - x_0$

In this case, we can integrate both sides to obtain

$x_0 \rightarrow @ t=0$

$$\int_{x(t_0)}^{x(t)} dx = \int_{t_0}^{t} f(t)\,dt$$

or

$$x(t) - x(t_0) = \int_{t_0}^{t} f(t)\,dt \tag{1.4.1}$$

Obviously you may need an integral table to solve some problems of this type, depending on the complexity of the function $f(t)$. The value of $x(t_0)$ is called the *initial condition*.

For example, the relation between the acceleration $a(t)$ and velocity $v(t)$ of an object is a differential equation, and we can obtain the velocity $v(t)$ with direct integration as follows:

$a = \frac{dv}{dt}$

$$\boxed{\frac{dv}{dt} = a(t),} \qquad v(t) - v(t_0) = \int_{t_0}^{t} a(t)\,dt$$

The relation between the velocity $v(t)$ and the displacement $x(t)$ is also a differential equation, and we can obtain the displacement $x(t)$ as follows:

$v = \frac{dx}{dt}$

$$\frac{dx}{dt} = v(t), \qquad x(t) - x(t_0) = \int_{t_0}^{t} v(t)\,dt$$

Suppose that the acceleration is $a(t) = 6t^2$, $t_0 = 0$, and the initial velocity is $v(0) = 5$. Then

$$v(t) - 5 = \int_0^t 6t^2\,dt = 2t^3$$

Thus, $v(t) = 5 + 2t^3$. If the initial displacement is $x(0) = 7$,

$$x(t) - 7 = \int_0^t \left(5 + 2t^3\right) dt = 5t + \frac{t^4}{2}$$

### 1.4.2 APPLICATION TO A PLANETARY LANDER

From basic physics recall that Newton's law of motion for a mass $m$ translating in one dimension $x$ is

$$m\frac{d^2m}{dt^2} = m\ddot{x} = \sum f_i$$

where $\ddot{x}$ is the acceleration and $\sum f_i$ is the algebraic sum of all forces acting in the $x$ direction.

For example, consider the problem of a rocket or planetary lander attempting to make a "soft" landing, which is defined as the lander having zero velocity when

$$\boxed{\Sigma f = m\ddot{x}}$$

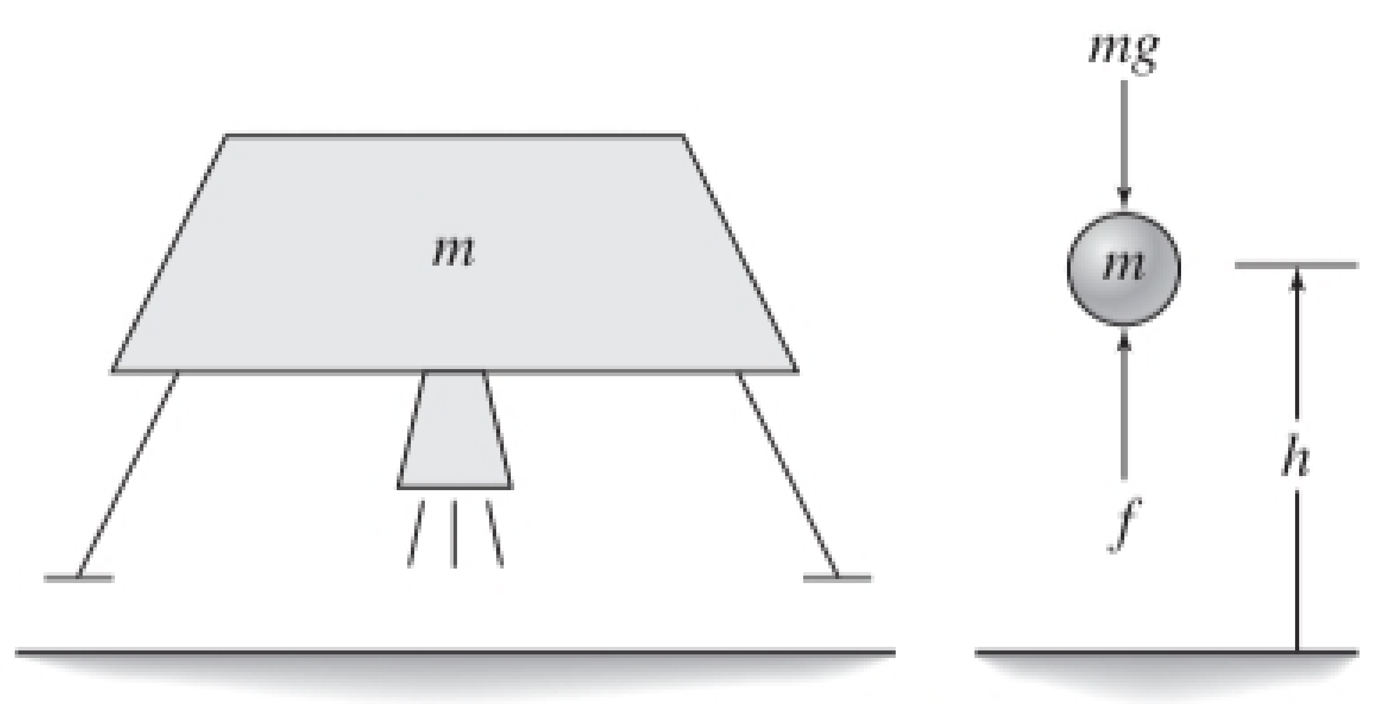



**Figure 1.4.1** Simple model of the vertical motion of a lander.

touchdown occurs at $h = 0$ (Figure 1.4.1). A simple representation of this situation is given by Newton's law:

$$m\ddot{h} = f - mg$$

where $mg$ is the lander weight.

**Soft Landing with Constant Thrust** We will use this model to answer the following question: Is it possible to achieve a soft landing using *constant* thrust? To answer this, we first simplify the problem, if possible, to analyze only the most important aspects of the problem. This is often hard to do, and experience definitely helps (which you hopefully will obtain from all of our examples!). So we will assume that the acceleration $g$ due to gravity is constant; this means that the lander does not have a great change in altitude. We neglect any atmospheric drag. Can you think of other simplifications being used in the following model?

Express the thrust as a function of the weight as follows: $f = \beta mg$, where $\beta$ is a constant that we need to compute. This lets us temporarily eliminate the mass from the model as follows:

$$m\ddot{h} = \beta mg - mg = (\beta - 1)mg = \alpha mg$$

or, after cancelling $m$, we obtain $\ddot{h} = \alpha g$, where $\alpha$ is a new constant: $\alpha = \beta - 1$. Integrating both sides once gives the velocity.

$$\dot{h}(t) = \alpha gt + \dot{h}(0) = \alpha gt - v_0$$

where $v_0 = -\dot{h}(0)$ is the initial vertical velocity *downward*. To achieve a soft landing, $\dot{h} = 0$, and the previous equation shows that the time to land $t_f$ is

$$t_f = \frac{v_0}{\alpha g} \tag{1.4.2}$$

Integrating the $\dot{h}$ expression gives

$$h(t) = \alpha g\frac{t^2}{2} - v_0 t + h_0$$

where $h_0$ is the initial height. Substituting $t = t_f$ into this equation gives

$$h(t_f) = \frac{\alpha g}{2}\left(\frac{v_0}{\alpha g}\right)^2 - \frac{v_0^2}{\alpha g} + h_0 = -\frac{v_0^2}{2\alpha g} + h_0 = 0$$

This gives

$$\alpha = \frac{v_0^2}{2gh_0} \tag{1.4.3}$$

Thus, we see that the value of $\alpha$ is fixed by $g$, the initial velocity $v_0$, and the initial height $h_0$. The required thrust $f = (\alpha + 1)mg$ will always be positive, as required by the physics of rocket engines. From (1.4.2) and (1.4.3) we see that the time to land is fixed by $v_0$ and $h_0$.

The mathematics does not realize that the height cannot be negative, so we should check for that possibility by checking to see if $h(t)$ passes through 0 any time before $t_f$. Setting $h(t)$ equal to zero gives

$$0 = \alpha g \frac{t^2}{2} - v_0 t + h_0$$

From the quadratic formula and using (1.4.3), we have

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 2\alpha g h_0}}{\alpha g} = \frac{v_0}{\alpha g}$$

which is the time to land, $t_f$. Thus, $h(t) \geq 0$.

**EXAMPLE 1.4.1** Constant Thrust Example

**■ Problem**

Suppose that $m = 2500$ kg and $g = 2$ m/s$^2$ (which indicates a planetary body smaller than the Earth). Suppose also that the initial downward velocity is $v_0 = 5$ m/s and the initial height is $h_0 = 500$ m. Find the thrust required for a soft landing and the time to land.

**■ Solution**

From (1.4.3) we have

$$\alpha = \frac{25}{2(2)(500)} = 0.0125$$

Thus, the thrust required is $f = \beta mg = (\alpha + 1)mg = 1.0125(2500)(2) = 5062.5$ N. The time to land is found from (1.4.2) to be $t_f = 5/[(0.0125)(2)] = 200$ s.

Note some of the other simplications made in obtaining this result. We assumed the lander mass is constant, but in fact it decreases with time due to fuel consumption. We also assumed purely vertical motion, whereas the lander could be descending from orbit. We also assumed that $g$ is constant, whereas in fact $g$ varies with altitude (but this is a good assumption when the altitude is only a few hundred meters).

**Linear Velocity Profile** We approached the lander problem by assuming that the thrust is constant. We could have approached the problem by assuming that the velocity $\dot{h}(t)$ is a certain function, called a *velocity profile*. For example, we could have assumed that the velocity is a linear function of time; that is,

$$\dot{h}(t) = at + b$$

where $a$ and $b$ are constants to be found. Differentiating this gives the acceleration $\ddot{h} = a$, and substituting this into the equation of motion gives

$$m\ddot{h} = ma = f - mg$$

This shows that $f = m(a + g)$, which is constant. Thus, the assumed linear velocity profile gives the same results as found with the constant thrust assumption.

**Quadratic Velocity Profile** Using constant thrust enables a soft landing to be achieved but we cannot specify the time of flight $t_f$. Noting that constant thrust results in a linear velocity profile, we can try another velocity profile that may let us also set a desired flight time. So we try a quadratic profile of the form

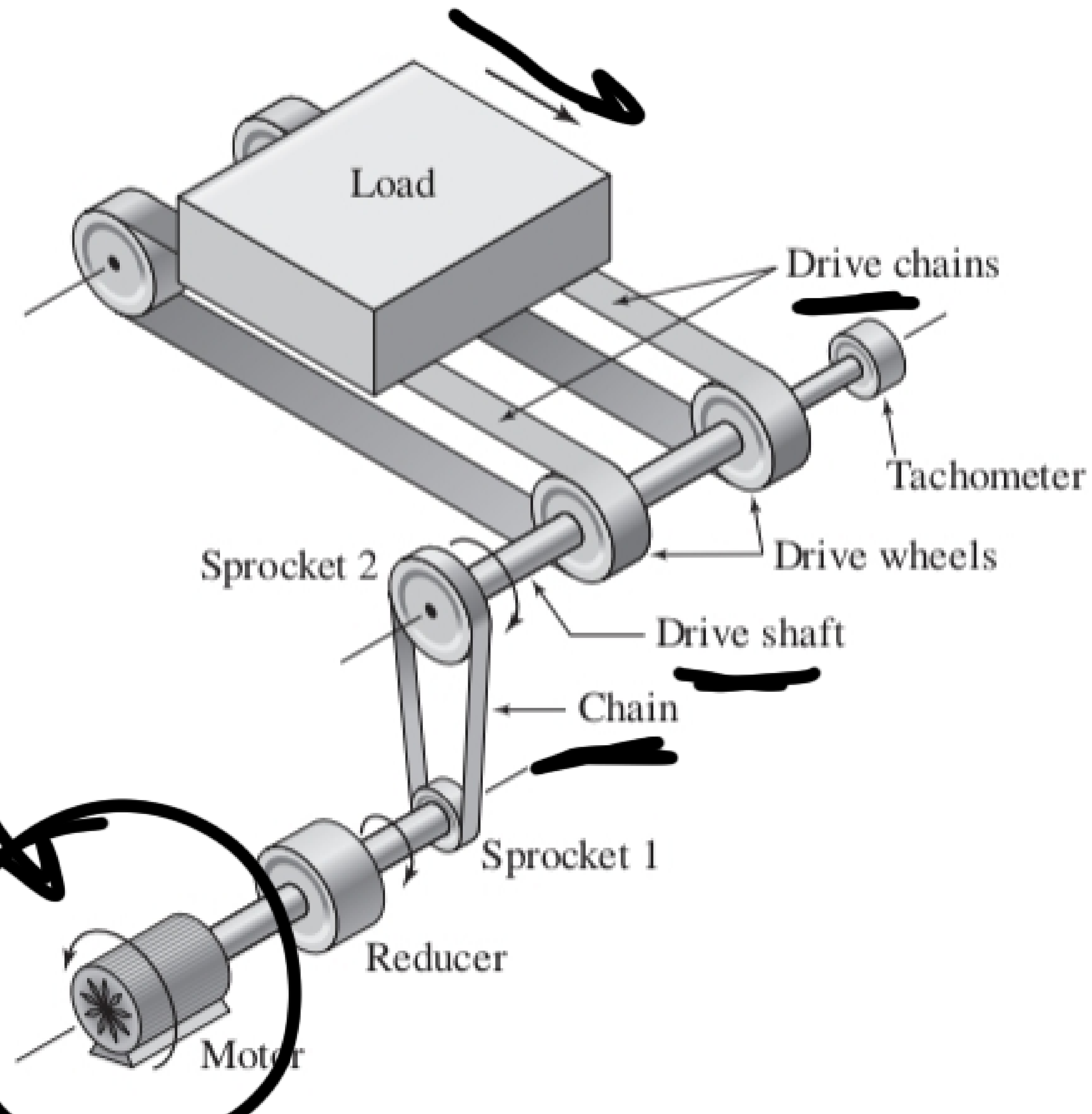
$$\dot{h}(t) = at^2 + bt + c$$

where we can easily show that $c = -v_0$ and $\ddot{h} = 2at + b$. Substituting this into the equation motion shows that the thrust must now be a linear time function: $f = m(2at + b + mg)$. One of the chapter problems asks you to compute the required values of $a$ and $b$.

## 1.5 A CASE STUDY IN MOTION CONTROL

*Motion-control engineering* is a specialty within the field of automation and involves the design of systems for controlling position or speed of machines. The main components of such systems are a motion controller (such as electronics or a computer), a power amplifier, and an actuator or prime mover to do the work. Typical actuators include hydraulic or pneumatic pistons and electric motors. Motion control is widely used in robotics and computer numerically controlled (CNC) machine tools, and in assembly lines for textiles, packaging, and printing. The term "motion controller" is also used to describe a type of video game controller used to provide input, but this application is not discussed here.

Various kinds of mechanical components are used with the actuator to achieve the desired motion. These include gears, power screws, and belts or chains. Figure 1.5.1 shows a typical motion-control system designed to translate an object (called the "load") over a certain distance with a desired speed. In Chapter 3 we will return to this system to analyze how well it functions. We will show how to model it simply, as if only a single rotational object (called the "equivalent load" or "reflected load") is connected to the motor shaft.



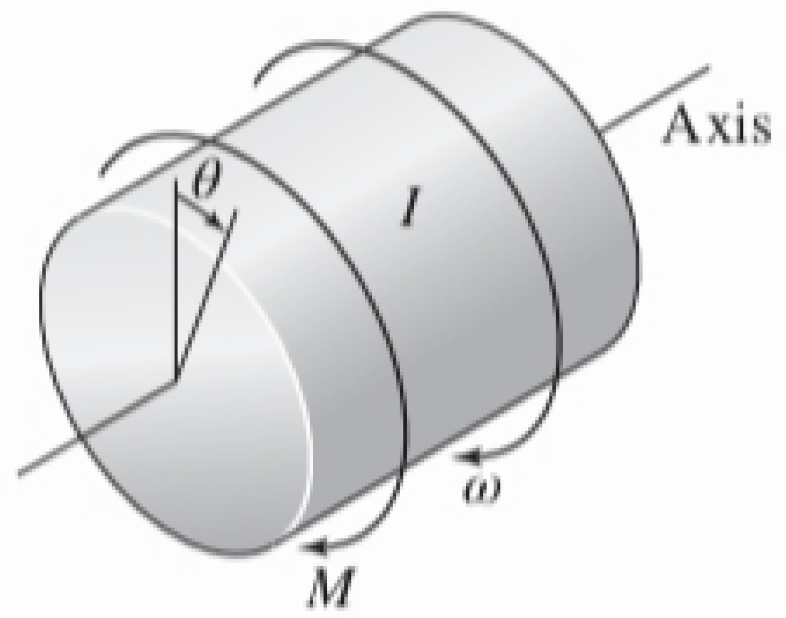
$\boxed{T = I\alpha}$

If we consider the dynamics of a rigid body whose motion is constrained to allow only rotation about an axis through a nonaccelerating point, then from basic physics we know that the equation of motion is

$$I\frac{d\omega}{dt} = I\dot{\omega} = M$$

where $\omega$ is the angular velocity of the mass about an axis through a point fixed in an inertial reference frame and attached to the body (or the body extended), $I$ is the mass moment of inertia of the body about that point, and $M$ is the sum of the moments applied to the body about that point. This situation is illustrated in Figure 1.5.2. The angular displacement is $\theta$, always measured in radians, and $\omega = d\theta/dt = \dot{\theta}$, whose units are radians per unit time. The term *torque* and thus the symbol $T$ are often used instead of *moment* and $M$. When the context is unambiguous, we use the term *inertia* as an abbreviation for *mass moment of inertia*. In Chapter 3 we consider the case of more general rotation and show how $I$ is calculated for specific body shapes.

**Figure 1.5.2** An object rotating about a fixed axis.




**EXAMPLE 1.5.1** | Constant Torque Solution

**■ Problem**

Figure 1.5.3 illustrates a basic example of motion control where a motor supplies a torque $T$ to an object, called the "load," whose inertia is $I_L$. The rotating part of the motor is called the "armature" and has an inertia $I_m$ (in Chapter 6 we will study motor design and performance in more detail). If the shaft connecting the motor to the load does not twist, then the motor and load rotate at the same speed $\omega$ and the combined inertia of the system will be the sum: $I = I_m + I_S + I_L$ where $I_m$, $I_S$, and $I_L$ are the inertias of the motor, shaft, and load, respectively. The equation of motion is given by

$\Sigma I = I_m + I_s + I_L$

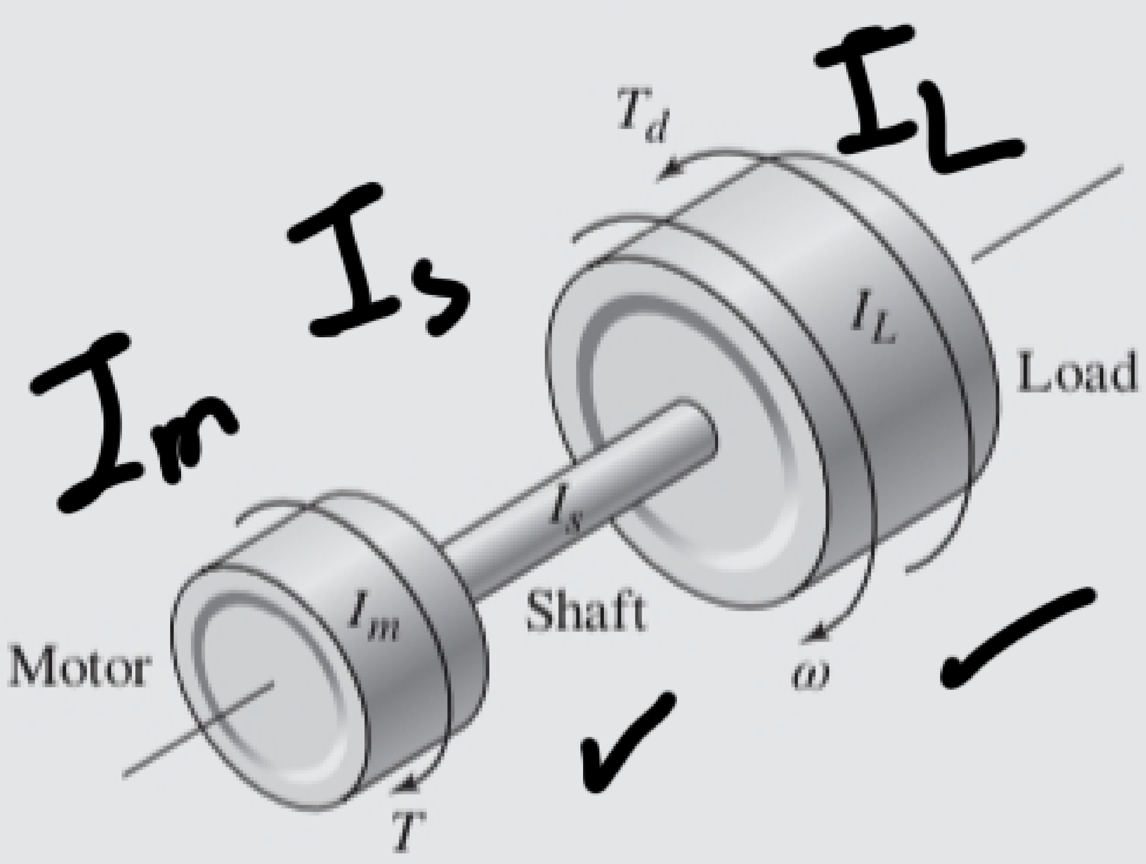
$$(I_m + I_S + I_L)\dot{\omega} = T \quad \text{or} \quad I\dot{\omega} = T \tag{1}$$

$\alpha = \dot{\omega}$

$\omega = \dot{\theta}$

Obtain the speed and displacement as functions of time.

**Figure 1.5.3** Simple model of a mechanical drive with a rotational load.




**■ Solution**

If the motor torque is constant, we can solve the equation for the speed and the angular displacement as functions of time, as follows:

$$\omega(t) = \int_0^t \dot{\omega}(t)\,dt + \omega(0) = \int_0^t \frac{T}{I}\,dt + \omega(0) = \frac{T}{I}t + \omega(0)$$

$$\theta(t) = \int_0^t \omega(t)\,dt + \theta(0) = \int_0^t \left[\frac{T}{I}t + \omega(0)\right]dt + \theta(0) = \frac{T}{I}\frac{t^2}{2} + \omega(0)t + \theta(0)$$

If the system starts from rest at $\theta(0) = 0$, then $\omega(0) = 0$ and

$$\omega(t) = \frac{T}{I}t, \qquad \theta(t) = \frac{T}{I}\frac{t^2}{2}$$

So a constant applied torque produces a linear velocity profile and a parabolic displacement profile.

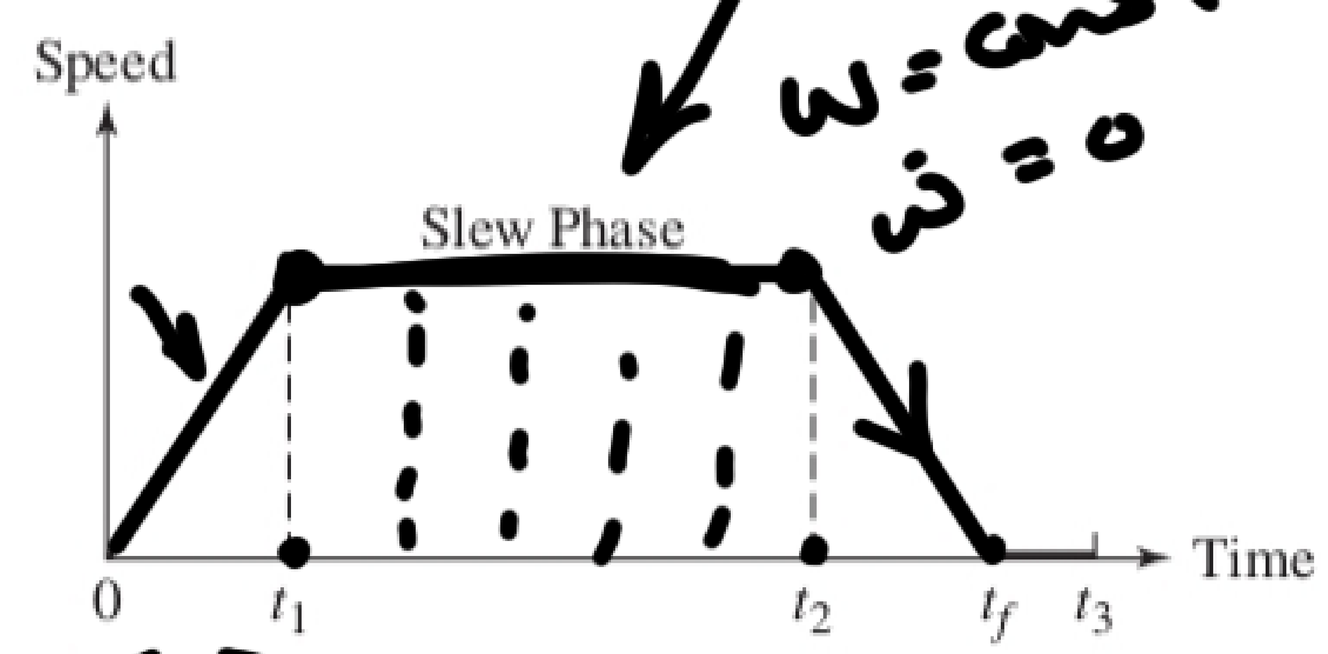
**Trapezoidal Velocity Profile** In many motion-control applications we want to accelerate the system to a desired speed and run it at that speed for some time before decelerating to a stop, as illustrated by the trapezoidal speed profile shown in Figure 1.5.4. The constant-speed phase of the profile is called the *slew* phase. An example requiring a slew phase of specified duration and speed is a conveyor system in a manufacturing line, in which the item being worked on must move at a constant speed for a specified time while some process is applied, such as painting or welding. The speed profile can be specified in terms of rotational motor speed $\omega$ or in terms of the translational speed $v$ of a load.

In other applications, the duration and speed of the slew phase is not specified, but we want the system to move (or rotate) a specified distance (or angle) by the end of one cycle. An example of this is a robot arm link that must move through a specified angle. The acceleration and deceleration times, and the duration and speed of the slew phase must be computed to achieve the desired distance or angle. Often the profile is followed by a zero-speed phase to enable the motor to cool before beginning the next cycle. The profile in Figure 1.5.4 assumes that the system begins and ends at rest. Note also that $t_f - t_2$ must equal $t_1$ if the profile is symmetric.

Available motor torque is aways limited, and the manufacturer will specify the maximum available torque $T_{max}$, which is the torque the motor can provide for a short time. The manufacturer will also state the *rated continuous torque*, also called *duty cycle torque*. This is the torque the motor can provide continuously without being damaged, for example, by overheating. The average torque required for a specific application is computed by the *root mean square* method and is denoted $T_{rms}$. The profile must not require an average torque greater than the rated continuous torque. The root mean square average is defined as

$$T_{rms} = \sqrt{\frac{1}{t_f}\int_0^{t_f} T^2(t)\,dt} \tag{1.5.1}$$

Thus, because of the squared torque, the rms average gives equal weight to positive and negative values of torque.




$\omega = $ const.
$\dot{\omega} = 0$

0 0.5 2.5 3

**Figure 1.5.4** A trapezoidal speed profile.

**EXAMPLE 1.5.2** Maximum Versus Average Required Torque

$kg \cdot m^2$

**■ Problem**

Suppose that $I_m = 0.003$, $I_S = 0.001$, and $I_L = 0.006$ so that $I = I_m + I_S + I_L = 0.01$ kg·m² for the system shown in Figure 1.5.3. Compute how much motor torque is required to achieve the velocity profile shown in Figure 1.5.4, where $t_1 = 0.5$ s, $t_2 = 2.5$ s, and $t_f = 3$ s, and the slew speed is 150 rad/s.

**■ Solution**

From Figure 1.5.4, we obtain the required angular acceleration from the slope of the velocity curve. This gives

$\boxed{T = I\alpha}$

$$\alpha = \quad \dot{\omega} = \begin{cases} \frac{150}{0.5} = 300 & 0 \le t \le 0.5 \\ 0 & 0.5 < t < 2.5 \\ -\frac{150}{0.5} = -300 & 2.5 \le t \le 3 \end{cases} \qquad \frac{\omega}{t}$$

Thus, the required torque is

$t_f: 3$ sec

$$T = I\dot{\omega} = \begin{cases} 3 & 0 \le t \le 0.5 \\ 0 & 0.5 < t < 2.5 \\ -3 & 2.5 \le t \le 3 \end{cases}$$

So the maximum required torque is 3 N·m.

The rms average torque is calculated as follows:

$$T_{rms} = \sqrt{\frac{1}{3}\left[\int_0^{0.5} 3^2 dt + \int_{0.5}^{2.5} 0^2 dt + \int_{2.5}^{3} (-3)^2 dt\right]} = \sqrt{\frac{1}{3}\left[3^2(0.5) + 0 + (-3)^2(0.5)\right]}$$

$$= \sqrt{3} = 1.7321 \text{ N·m}$$

Note that $T_{rms}$ is obviously never greater than the maximum torque.

**EXAMPLE 1.5.3** Friction Effects

**■ Problem**

Suppose the load in Example 1.5.2 is acted on by a torque $T_d = 2$ N·m caused by Coulomb (dry) friction. Recalulate the maximum and rms torques.

**■ Solution**

The equation of motion is now

$$I\dot{\omega} = T - T_d \qquad \omega > 0$$

Thus,

$T = I\alpha + T_f$

$$T = I\dot{\omega} + T_d = \begin{cases} 3 + 2 = 5 & 0 \le t \le 0.5 \\ 2 & 0.5 < t < 2.5 \\ -3 + 2 = -1 & 2.5 \le t \le 3 \end{cases}$$

So the maximum required torque is 5 and the rms torque is

$$T_{rms} = \sqrt{\frac{1}{3}\left[(5)^2(0.5) + (2)^2(2) + (-1)^2(0.5)\right]} = 2.6458 \text{ N·m}$$

## GENERAL RESULTS FOR A TRAPEZOIDAL VELOCITY PROFILE

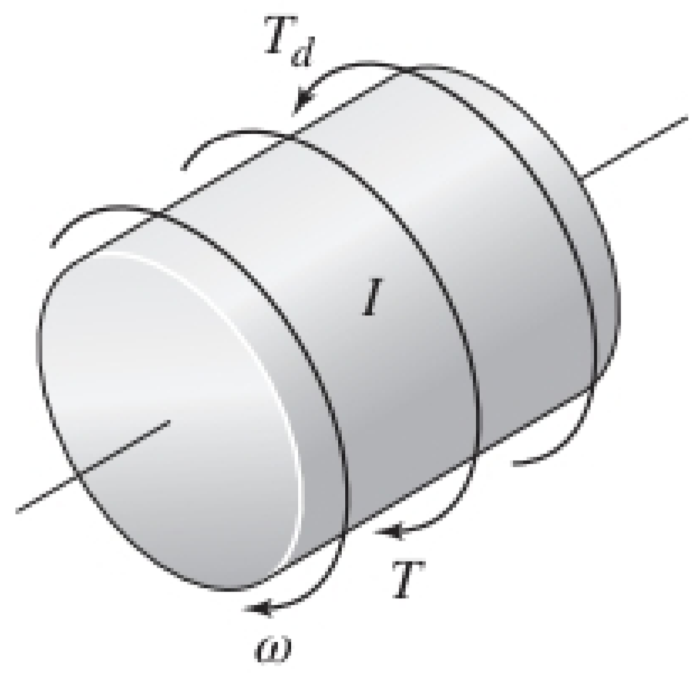



**Figure 1.5.5** A rotational system with a single inertia.

The results of the two previous examples can be generalized as follows. The equation of motion for the system in Figure 1.5.5 is

$$I\dot{\omega} = T - T_d \tag{1.5.2}$$

where $I$ is the system inertia at the motor shaft, $T$ is the motor torque, and $T_d$ is a disturbance torque such as that caused by friction.

**Maximum Motor Speed for a Specified Displacement** Suppose that the application requires that the load rotate through a specified displacement $\theta_f$ in the specified time $t_f$. For this application the trapezoidal profile will specify the *angular* velocity rather than a translational velocity. Thus, the total angular displacement $\theta_f$ is the area under the trapezoidal profile. Assuming that $\theta(0) = 0$ and because $t_f - t_2 = t_1$, we have

$$\theta(t_f) = \theta_f = \int_0^{t_f} \omega(t)dt = 2\left(\frac{1}{2}\omega_{max}t_1\right) + \omega_{max}(t_2 - t_1) = \omega_{max}t_2$$

Thus, the maximum required motor speed is given by

$$\omega_{max} = \frac{\theta_f}{t_2} \tag{1.5.3}$$

So if we specify the displacement $\theta_f$ and the time $t_2$, the maximum required speed is determined from this equation.

**Maximum Motor Torque for a Specified Displacement** Let $\alpha$ denote the angular acceleration $\dot{\omega}$. The maximum required angular acceleration for the trapezoidal profile is

$$\alpha_{max} = \frac{\omega_{max}}{t_1} = \frac{\theta_f}{t_1 t_2} \tag{1.5.4}$$

Using (1.5.2) through (1.5.4), the maximum required motor torque is given by

$$T_{max} = I\alpha_{max} + T_d = I\frac{\omega_{max}}{t_1} + T_d = I\frac{\theta_f}{t_1 t_2} + T_d \tag{1.5.5}$$

**RMS Motor Torque** The rms torque is calculated from (1.5.1) and (1.5.2):

$$T_{rms}^2 = \frac{1}{t_f}\int_0^{t_f} T^2(t)\,dt = \frac{1}{t_f}\int_0^{t_f} \left(I\alpha + T_d\right)^2 dt$$

or

$$T_{rms}^2 = \frac{1}{t_f}\int_0^{t_1} \left(I\alpha_{max} + T_d\right)^2 dt + \frac{1}{t_f}\int_{t_1}^{t_2} \left(0 + T_d\right)^2 dt + \frac{1}{t_f}\int_{t_2}^{t_f} \left(-I\alpha_{max} + T_d\right)^2 dt$$

This reduces to

$$T_{rms}^2 = \frac{1}{t_f}\left[2I^2\alpha_{max}^2 t_1 + T_d^2\left(t_1 + t_2\right)\right]$$

or, since $\alpha_{max} = \theta_f/t_1 t_2$ and $t_1 + t_2 = t_f$,

$$T_{rms} = \sqrt{\frac{2I^2\theta_f^2}{t_f t_1 t_2^2} + T_d^2} \tag{1.5.6}$$

**Table 1.5.1** Motor Requirements for Trapezoidal Speed Profile.

| | |
|---|---|
| Profile times $t_1$, $t_2$, $t_f$ | See Figure 1.5.4 |
| Motor displacement | $\theta_f$ = area under speed profile |
| Load torque felt at motor | $T_d$ |
| Maximum Speed | $\omega_{max} = \frac{\theta_f}{t_2}$ |
| Maximum Acceleration | $\alpha_{max} = \frac{\theta_f}{t_1 t_2}$ |
| Maximum Torque | $T_{max} = I\frac{\theta_f}{t_1 t_2} + T_d$ |
| RMS Average Torque | $T_{rms} = \sqrt{\frac{2I^2\theta_f^2}{t_f t_1 t_2^2} + T_d^2}$ |

These equations are summarized in Table 1.5.1 and are used to compute the motor requirements for the trapezoidal speed profile for an application where the total angular displacement is specified. To determine whether a given motor will be satisfactory, the calculated values of $\omega_{max}$, $T_{max}$, and $T_{rms}$ are compared to the manufacturer's data on maximum speed, peak torque, and rated continuous torque.

**EXAMPLE 1.5.4**

### Motor Calculations for a Trapezoidal Profile

**■ Problem**

A certain system like that shown in Figure 1.5.3 has a load inertia $I_L = 6 \times 10^{-3}$ kg·m², a shaft inertia of $I_S = 5\times 10^{-4}$ kg·m², and a motor inertia of $I_m = 10^{-3}$ kg·m². A friction torque of 2 N·m acts on the load. It is desired to rotate the load through $180° = \pi$ rad in one-half second from start to stop. The load should be rotated at a constant speed for 0.4 s during the slew phase. Compute the motor's maximum speed, the maximum required torque, and the rms average torque.

**■ Solution**

The total load displacement is given as $\theta_f = \pi$ rad. We also have $t_f = 0.5$ s and a slew time of $t_2 - t_1 = 0.4$ s. Using a symmetrical velocity profile requires that $t_1 = 0.05$ and $t_2 = 0.45$. The total inertia is $I = 6 \times 10^{-3} + 5 \times 10^{-4} + 10^{-3} = 7.5 \times 10^{-3}$. From Table 1.5.1 we have $\omega_{max} = \pi/0.4 = 6.9813$ rad/s or $60\omega_{max}/(2\pi) = 66.6667$ rpm. So the load will rotate at that speed for 0.4 s.

We are given that $T_d = 2$ N·m. So the maximum torque required is $T_{max} = 3.0472$ N·m and the rms average torque is $T_{rms} = 2.0541$ N·m.

**Lead Screws** A lead screw or power screw (Figure 1.5.6) is used to convert the rotational motion of a motor into linear motion of a load mounted on the nut. They are easy

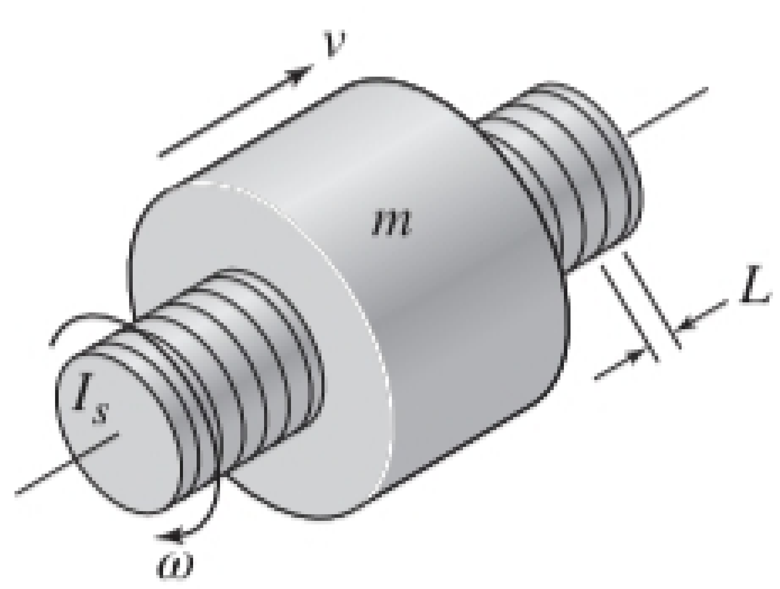



**Figure 1.5.6** A lead screw.

to design and manufacture and can carry large loads with high precision. However, because of the large contact area between the sliding surfaces of the nut and screw, they can have large frictional energy losses.

As will be discussed in more detail in Chapter 3, we can model the effect of the translating load by computing an equivalent inertia. We do this by first obtaining an expression for the kinetic energy of the load. If the distance between two adjacent threads is $L$ (called the *screw lead*), then a screw rotation through an angle $\theta$ in radians will produce a load translation $x$, where $x = L\theta/(2\pi)$, and a translational velocity of $\dot{x} = v = L\omega/(2\pi)$, where $\omega = \dot{\theta}$. Thus, the kinetic energy KE of the load is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{L\omega}{2\pi}\right)^2$$

whereas the kinetic energy of a rotating inertia $I_L$ is

$$KE = \frac{1}{2}I_L\omega^2$$

Equating the two expressions for kinetic energy shows that

$$KE = \frac{1}{2}m\left(\frac{L\omega}{2\pi}\right)^2 = \frac{1}{2}I_L\omega^2$$

Solving for $I_L$ gives the expression for the equivalent inertia of the load:

$$I_L = \frac{mL^2}{4\pi^2} \tag{1.5.7}$$

This is the inertia that the motor "feels" due to the load.

The total inertia felt by the motor is the sum of the motor's own inertia $I_m$, the equivalent load inertia $I_L$, and the inertia of the screw, which we obtain by modeling the screw as a cylindrical rod. The formula for the screw inertia is

$$I_S = \frac{1}{2}m_S R^2 \tag{1.5.8}$$

where $m_S$ and $R$ are the mass and radius of the cylinder.

## Sizing a Motor for a Lead Screw — EXAMPLE 1.5.5

**■ Problem**

Neglect friction and estimate the torque required to accelerate a 100-kg load mass from rest to 0.2 m/s in 0.1 s, using a steel lead screw of length 1.1 m and radius 0.013 m. The screw lead is $L = 0.005$ m/rev. The motor inertia is $2 \times 10^{-4}$ kg·m$^2$. Take the density of steel to be 7700 kg/m$^3$.

$\boxed{m = \rho V}$

**■ Solution**

The equivalent inertia of the load mass is, from (1.5.7),

$$I_L = \frac{100(0.005)^2}{4\pi^2} = 6.3326 \times 10^{-5}$$

The mass of the screw is

$$m_S = \rho V = 7700\pi(0.013)^2(1.1) = 4.4970 \text{ kg}$$

and the inertia of the screw is

$$I_S = \frac{1}{2}4.4970(0.013)^2 = 3.8 \times 10^{-4}$$

 $\alpha = \dot{\omega}$ 

$T = I\alpha$

So the total system inertia is

$$I = I_m + I_L + I_S = 2 \times 10^{-4} + 6.3326 \times 10^{-5} + 3.8 \times 10^{-4} = 6.4333 \times 10^{-4}\ \text{kg·m}^2$$

From the equation of motion, the required torque is $T = I\dot{\omega}$. The linear displacement $x$ of the mass is related to the rotation angle $\theta$ of the screw by $x = L\theta/2\pi$. Thus, the linear acceleration $\ddot{x}$ is related to the angular acceleration $\ddot{\theta} = \dot{\omega}$ by $\dot{\omega} = 2\pi\ddot{x}/L$.

The given linear acceleration is $\ddot{x} = (0.2 - 0)/0.1 = 2\ \text{m/s}^2$. Thus,

$$\dot{\omega} = \frac{2\pi(2)}{0.005} = 2513\ \ \text{rad/s}^2$$

Thus, the required torque is $T = 6.4333 \times 10^{-4}(2513) = 1.6168$ N·m.

The calculations given in Table 1.5.1 can also be applied to a lead screw system using a trapezoidal translational velocity profile, as long as the specified translational displacement, speed, and acceleration are first converted to their equivalent rotational quantities using the screw lead $L$.

**Further Studies in Motion Control** In Chapter 3 we will continue our study of motion control by showing how various mechanical elements such as gears and belts can be modeled with an equivalent inertia. Then in Chapter 6 we will show how an electric motor can be selected to provide the required speed and torque. Sometimes external effects such as friction are difficult to predict, and Chapter 10 shows how to design a computer control system to counteract such effects.